Journal of Global Optimization (2006) 36: 237–282
DOI 10.1007/s10898-006-9008-1

Generalized (η, ρ)-Invex Functions and Semiparametric Duality Models for Multiobjective Fractional Programming Problems Containing Arbitrary Norms

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(Received 31 May 2005; accepted in revised form 6 February 2006; Published online 14 June 2006)

Abstract. In this paper, we construct several semiparametric duality models and prove appropriate duality theorems under various generalized (η, ρ)-invexity assumptions for a multiobjective fractional programming problem involving arbitrary norms.

AMS Subject Classification. 49N15, 90C26, 90C29, 90C30, 90C32

Key words: Arbitrary norms, Generalized (η, ρ)-invex functions, Multiobjective fractional programming, Semiparametric duality models

1. Introduction

In this paper, we present a fairly large number of semiparametric duality results under a variety of generalized (η, ρ)-invexity conditions for the following multiobjective fractional programming problem involving nondifferentiable functions:

$$(P) \qquad \text{Minimize } \left(\frac{f_1(x) + \|A_1 x\|_{a(1)}}{g_1(x) - \|B_1 x\|_{b(1)}}, \dots, \frac{f_p(x) + \|A_p x\|_{a(p)}}{g_p(x) - \|B_p x\|_{b(p)}} \right)$$

subject to

$$G_j(x) + \|C_j x\|_{c(j)} \leqq 0, \ j \in \underline{q}, \quad H_k(x) = 0, \ \ k \in \underline{r}, \ \ x \in X,$$

where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), $f_i, g_i, i \in \underline{p} \equiv \{1, 2, \dots, p\}, G_j, \ j \in \underline{q}$, and $H_k, k \in \underline{r}$, are real-valued functions defined on X, for each $i \in \underline{p}$ and each $j \in \underline{q}$, A_i, B_i, and C_j are, respectively, $\ell_i \times n$, $m_i \times n$, and $n_j \times n$ matrices, $\|\cdot\|_{a(i)}$, $\|\cdot\|_{b(i)}$, and $\|\cdot\|_{c(j)}$ are arbitrary norms in $\mathbb{R}^{\ell_i}$, $\mathbb{R}^{m_i}$, and $\mathbb{R}^{n_j}$, respectively, and for each $i \in \underline{p}$, $g_i(x) - \|B_i x\|_{b(i)} > 0$ for all x satisfying the constraints of (P).

Several classes of static and dynamic optimization problems with multiple fractional objective functions have been the subject of intense investigations in the past few years, which have produced a number of sufficiency and duality results for these problems. Fairly extensive lists of references pertaining to various aspects of multiobjective fractional programming are available in [22–25]. For more information about the vast general area of multiobjective programming, the reader may consult [11, 16, 18, 20].

A close examination of these and other related sources will readily reveal the fact that so far multiobjective fractional programming problems containing arbitrary norms in their objective functions and constraints have not been studied in the area of multiobjective programming. In the present study, we shall formulate several semiparametric dual problems for (P) and establish numerous duality results under various generalized (η, ρ)-invexity conditions. These duality formulations are based on the necessary and sufficient efficiency criteria presented in [26].

The rest of this paper is organized as follows. In Section 2, we present a number of definitions and auxiliary results which will be needed in the sequel. In Section 3, we consider four duality models with somewhat limited constraint structures, and prove weak, strong, and strict converse duality theorems under two sets of conditions. In Section 4, we formulate another set of four duality models with much more flexible constraint structures which allow for a greater variety of generalized (η, ρ)-invexity hypotheses under which duality can be established. We continue our discussion of duality in Sections 5 and 6 where we use two partitioning schemes and construct eight generalized duality models and obtain several duality results under various generalized (η, ρ)-invexity assumptions. In fact, each one of these eight duality models is a family of dual problems for (P) whose members can easily be identified by appropriate choices of certain sets and functions. Finally, in Section 7, we summarize our main results and also point out some further research opportunities.

It is evident that all the duality results obtained for (P) are also applicable, when appropriately specialized, to the following ten classes of problems with multiple, fractional, and conventional objective functions, which are particular cases of (P):

$$\text{(P1)} \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \quad \left(f_1(x)+\|A_1x\|_{a(1)}, \dots, f_p(x)+\|A_px\|_{a(p)}\right);$$

$$\text{(P2)} \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \quad \frac{f_1(x)+\|A_1x\|_{a(1)}}{g_1(x)-\|B_1x\|_{b(1)}};$$

$$\text{(P3)} \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \quad f_1(x)+\|A_1x\|_{a(1)},$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in X : G_j(x) + \|C_j x\|_{c(j)} \leqq 0,\ j \in \underline{q},\ H_k(x) = 0,\ k \in \underline{r}\};$$

$$\text{(P4)} \qquad \text{Minimize} \left(\frac{f_1(x) + \langle x, P_1 x\rangle^{1/2}}{g_1(x) - \langle x, Q_1 x\rangle^{1/2}}, \ldots, \frac{f_p(x) + \langle x, P_p x\rangle^{1/2}}{g_p(x) - \langle x, Q_p x\rangle^{1/2}} \right)$$

subject to

$$G_j(x) + \langle x, R_j x\rangle^{1/2} \leqq 0, \quad j \in \underline{q}, \qquad H_k(x) = 0, \quad k \in \underline{r}, \quad x \in X,$$

where P_i, Q_i, $i \in \underline{p}$, and R_j, $j \in \underline{q}$, are $n \times n$ symmetric positive semidefinite matrices, $\langle u, v\rangle$ denotes the inner (scalar) product of the ν-dimensional vectors u and v, that is, $\langle u, v\rangle = \sum_{i=1}^{\nu} u_i v_i$, where u_i and v_i are the ith components of u and v, respectively, and for each $i \in \underline{p}$, $g_i(x) - \langle x, Q_i x\rangle^{1/2} > 0$ for all feasible solutions of (P4);

$$\text{(P5)} \qquad \underset{x \in \mathbb{G}}{\text{Minimize}} \left(f_1(x) + \langle x, P_1 x\rangle^{1/2}, \ldots, f_p(x) + \langle x, P_p x\rangle^{1/2} \right);$$

$$\text{(P6)} \qquad \underset{x \in \mathbb{G}}{\text{Minimize}}\ \frac{f_1(x) + \langle x, P_1 x\rangle^{1/2}}{g_1(x) - \langle x, Q_1 x\rangle^{1/2}};$$

$$\text{(P7)} \qquad \underset{x \in \mathbb{G}}{\text{Minimize}}\ f_1(x) + \langle x, P_1 x\rangle^{1/2},$$

where $\mathbb{G}$ is the feasible set of (P4), that is,

$$\mathbb{G} = \{x \in X : G_j(x) + \langle x, R_j x\rangle^{1/2} \leqq 0, \quad j \in \underline{q}, \qquad H_k(x) = 0, \quad k \in \underline{r}\};$$

$$\text{(P8)} \qquad \underset{x \in \mathbb{H}}{\text{Minimize}} \quad \left(f_1(x), \ldots, f_p(x) \right);$$

$$\text{(P9)} \qquad \underset{x \in \mathbb{H}}{\text{Minimize}} \quad \frac{f_1(x)}{g_1(x)};$$

$$\text{(P10)} \qquad \underset{x \in \mathbb{H}}{\text{Minimize}} \quad f_1(x),$$

where $\mathbb{H} = \{x \in X : G_j(x) \leqq 0, \quad j \in \underline{q}, \quad H_k(x) = 0, \quad k \in \underline{r}\}$.

The problems (P4), (P5), (P6), and (P7) are special cases of (P), (P1), (P2), and (P3), respectively, which are obtained by choosing $\|\cdot\|_{a(i)}, \|\cdot\|_{b(i)}, i \in \underline{p}$, and $\|\cdot\|_{c(j)}, j \in \underline{q}$, to be the ℓ_2-norm $\|\cdot\|_2$, and defining $P_i = A_i^T A_i$, $Q_i = B_i^T B_i$, $i \in \underline{p}$, and $R_j = C_j^T C_j$, $j \in \underline{q}$.

Since in most cases these results can easily be modified and restated for each one of the above ten problems, we shall not state them explicitly.

Optimization problems containing norms arise naturally in many areas of the decision sciences, applied mathematics, and engineering. They are encountered most frequently in facility location problems, approximation theory, and engineering design. A number of these problems have already been investigated in the related literature. Similarly, optimization problems involving square roots of positive semidefinite quadratic forms have arisen in stochastic programming, multifacility location problems, and portfolio selection problems, among others. A fairly extensive list of references pertaining to several aspects of these two classes of problems is given in [21].

2. Preliminaries

In this section we recall, for convenience of reference, the definitions of certain classes of generalized convex functions which will be needed in the sequel. We begin by defining an invex function which has been instrumental in creating a vast array of interesting and important classes of generalized convex functions.

DEFINITION 2.1. Let f be a real-valued differentiable function defined on X. Then f is said to be η-invex at y if there exists a function $\eta: X \times X \to \mathbb{R}^n$ such that for each $x \in X$,

$$f(x) - f(y) \geqq \langle \nabla f(y), \eta(x, y) \rangle,$$

where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)^T$ is the gradient of f at y and the superscript T denotes transposition; f is said to be η-invex on X if the above inequality holds for all $x, y \in X$.

From this definition it is clear that every real-valued differentiable convex function is invex with $\eta(x, y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson [5] who showed that for a nonlinear programming problem of the form

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leqq 0, \quad i \in \underline{m}, \quad x \in \mathbb{R}^n,$$

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \underline{m}$, are invex with respect to the same function $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush–Kuhn–Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [2] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasiinvex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been generalized in several directions. For our present purposes, we shall need a simple extension of invexity, namely, ρ-invexity which was originally defined in [8].

Let η be a function from $X \times X$ to $\mathbb{R}^n$, and let h be a real-valued differentiable function defined on X.

DEFINITION 2.2. The function h is said to be (strictly) (η, ρ)-invex at x^* if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$h(x) - h(x^*)(>) \geqq \langle \nabla h(x^*), \eta(x, x^*) \rangle + \rho \|x - x^*\|^2.$$

DEFINITION 2.3. The function h is said to be (prestrictly) (η, ρ)-*quasiinvex* at $x^* \in X$ if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$h(x)(<) \leqq h(x^*) \;\Rightarrow\; \langle \nabla h(x^*), \eta(x, x^*) \rangle \leqq -\rho \|x - x^*\|^2.$$

DEFINITION 2.4. The function h is said to be (strictly) (η, ρ)-*pseudoinvex* at $x^* \in X$ if there exists $\rho \in \mathbb{R}$ such that for each $x \in X (x \neq x^*)$,

$$\langle \nabla h(x^*), \eta(x, x^*) \rangle \geqq -\rho \|x - x^*\|^2 \Rightarrow h(x)(>) \geqq h(x^*).$$

From the above definitions it is clear that if h is (η, ρ)-invex at x^*, then it is both (η, ρ)-quasiinvex and (η, ρ)-pseudoinvex at x^*, if h is (η, ρ)-quasiinvex at x^*, then it is prestrictly (η, ρ)-quasiinvex at x^*, and if h is strictly (η, ρ)-pseudoinvex at x^*, then it is (η, ρ)-quasiinvex at x^*.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, (η, ρ)-pseudoinvexity can be defined in the following equivalent way: The function h is said to be (η, ρ)-pseudoinvex at x^* if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$h(x) < h(x^*) \Rightarrow \langle \nabla h(x^*), \eta(x, x^*) \rangle < -\rho \|x - x^*\|^2.$$

The concept of ρ-invexity has been extended in many ways, and various types of generalized ρ-invex functions have been utilized for establishing a variety of sufficient optimality criteria and duality relations for several classes of nonlinear programming problems. For more information about invex functions, the reader may consult [1–4, 6, 10, 12, 14, 17], and for recent surveys of these and related functions, the reader is referred to [9, 15].

In the remainder of this section, we recall a set of necessary efficiency conditions for (P) given in [26] which will play an important role in the construction and analysis of the dual problems that will be discussed in this paper. We begin by introducing a consistent notation for vector inequalities. For $a, b \in \mathbb{R}^m$, the following order notation will be used: $a \geqq b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$; $a \geqslant b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$, but $a \neq b$; $a > b$ if and only if $a_i > b_i$ for all $i \in \underline{m}$; and $a \ngeqslant b$ is the negation of $a \geqslant b$.

Consider the multiobjective problem

$$(P^*) \qquad \underset{x\in\mathcal{X}}{\text{Minimize}}\, F(x) = (F_1(x), \ldots, F_p(x)),$$

where F_i, $i \in \underline{p}$, are real-valued functions defined on the set $\mathcal{X}$.

An element $x^\circ \in \mathcal{X}$ is said to be an *efficient* (*Pareto optimal, nondominated, noninferior*) solution of (P*) if there exists no $x \in \mathcal{X}$ such that $F(x) \leqslant F(x^\circ)$.

THEOREM 2.1 [26]. *Let x^* be a normal efficient solution of (P) (i.e., an efficient solution of (P) at which a suitable constraint qualification holds) and assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are differentiable at x^*. Then there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that*

$$\sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\}$$
$$+ \sum_{j=1}^{q} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \tag{2.1}$$

$$v_j^* [G_j(x^*) + \|C_j x^*\|_{c(j)}] = 0, \quad j \in \underline{q}, \tag{2.2}$$

$$\|\alpha^{*i}\|^*_{a(i)} \leqq 1, \quad \|\beta^{*i}\|^*_{b(i)} \leq 1, \quad i \in \underline{p}, \tag{2.3}$$

$$\|\gamma^{*j}\|^*_{c(j)} \leqq 1, \quad j \in \underline{q}, \tag{2.4}$$

$$\langle \alpha^{*i}, A_i x^* \rangle = \|A_i x^*\|_{a(i)}, \quad \langle \beta^{*i}, B_i x^* \rangle = \|B_i x^*\|_{b(i)}, \quad i \in \underline{p}, \tag{2.5}$$

$$\langle \gamma^{*j}, C_j x^* \rangle = \|C_j x^*\|_{c(j)}, \quad j \in \underline{q}, \tag{2.6}$$

where $U = \{u \in \mathbb{R}^p : u > 0, \sum_{i=1}^p u_i = 1\}$, for each $i \in \underline{p}$, $N_i(x^*) = f_i(x^*) + \|A_i x^*\|_{a(i)}$, $D_i(x^*) = g_i(x^*) - \|B_i x^*\|_{b(i)}$, and $\|\cdot\|^*_a$ is the dual of the norm $\|\cdot\|_a$, that is, $\|\delta\|^*_a = \max_{\|\xi\|_a = 1} |\langle \delta, \xi \rangle|$.

The form and contents of the necessary efficiency conditions given in the above theorem along with the semiparametric sufficient efficiency conditions presented in [26] provide clear guidelines for formulating numerous duality models for (P). The rest of this paper is devoted to investigating various types of dual problems for (P).

In the remainder of this paper, we shall assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are differentiable on the open set X.

3. Duality Model I

In this section, we consider a dual problem with a relatively simple constraint structure and prove weak, strong, and strict converse duality theorems under (η, ρ)-invexity conditions. More general duality models for (P) will be discussed in the subsequent sections.

Consider the following four problems:

$$\text{(CI)} \quad \text{Maximize}\left(\frac{f_1(y)+\|A_1 y\|_{a(1)}}{g_1(y)-\|B_1 y\|_{b(1)}}, \ldots, \frac{f_p(y)+\|A_p y\|_{a(p)}}{g_p(y)-\|B_p y\|_{b(p)}}\right)$$

subject to

$$\sum_{i=1}^{p} u_i\{D_i(y)[\nabla f_i(y)+A_i^T\alpha^i]-N_i(y)[\nabla g_i(y)-B_i^T\beta^i]\} + \sum_{j=1}^{q} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k=1}^{r} w_k\nabla H_k(y)=0, \tag{3.1}$$

$$\sum_{j=1}^{q} v_j[G_j(y)+\|C_j y\|_{c(j)}]+\sum_{k=1}^{r} w_k H_k(y) \geqq 0, \tag{3.2}$$

$$\|\alpha^i\|_{a(i)}^* \leqq 1, \quad \|\beta^i\|_{b(i)}^* \leqq 1, \quad i \in \underline{p}, \tag{3.3}$$

$$\|\gamma^j\|_{c(j)}^* \leqq 1, \quad j \in \underline{q}, \tag{3.4}$$

$$\langle \alpha^i, A_i y\rangle = \|A_i y\|_{a(i)}, \quad \langle \beta^i, B_i y\rangle = \|B_i y\|_{b(i)}, \quad i \in \underline{p}, \tag{3.5}$$

$$\langle \gamma^j, C_j y\rangle = \|C_j y\|_{c(j)}, \quad j \in \underline{q}, \tag{3.6}$$

$$y \in X,\ u \in U, \quad v \in \mathbb{R}_+^q,\ w \in \mathbb{R}^r,\ \alpha^i \in \mathbb{R}^{\ell_i},\ \beta^i \in \mathbb{R}^{m_i},\ i \in \underline{p},\ \gamma^j \in \mathbb{R}^{n_j},\ j \in \underline{q}, \tag{3.7}$$

where for each $i \in \underline{p}$, $N_i(y)$ and $D_i(y)$ are as defined in Theorem 2.1;

$$(\tilde{\text{CI}}) \quad \text{Maximize}\left(\frac{f_1(y)+\|A_1 y\|_{a(1)}}{g_1(y)-\|B_1 y\|_{b(1)}}, \ldots, \frac{f_p(y)+\|A_p y\|_{a(p)}}{g_p(y)-\|B_p y\|_{b(p)}}\right)$$

subject to (3.2)–(3.7) and

$$\Big\langle \sum_{i=1}^{p} u_i\{D_i(y)[\nabla f_i(y) + A_i^T \alpha^i] - N_i(y)[\nabla g_i(y) - B_i^T \beta^i]\} + \sum_{j=1}^{q} v_j[\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x, y)\Big\rangle \geqq 0 \quad \text{for all } x \in \mathbb{F}, \tag{3.8}$$

where η is a function from $X \times X$ to $\mathbb{R}^n$;

$$\text{(DI)} \quad \text{Maximize} \left(\frac{f_1(y) + \langle \alpha^1, A_1 y\rangle}{g_1(y) - \langle \beta^1, B_1 y\rangle}, \ldots, \frac{f_p(y) + \langle \alpha^p, A_p y\rangle}{g_p(y) - \langle \beta^p, B_p y\rangle} \right)$$

subject to

$$\sum_{i=1}^{p} u_i\{D_i^{\circ}(y, \beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^{\circ}(y, \alpha)[\nabla g_i(y) - B_i^T \beta^i]\} + \sum_{j=1}^{q} v_j[\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k=1}^{r} w_k \nabla H_k(y) = 0, \tag{3.9}$$

$$\sum_{j=1}^{q} v_j[G_j(y) + \langle \gamma^j, C_j y\rangle] + \sum_{k=1}^{r} w_k H_k(y) \geqq 0, \tag{3.10}$$

$$\|\alpha^i\|_{a(i)}^* \leqq 1, \qquad \|\beta^i\|_{b(i)}^* \leqq 1, \quad i \in \underline{p}, \tag{3.11}$$

$$\|\gamma^j\|_{c(j)}^* \leqq 1, \quad j \in \underline{q}, \tag{3.12}$$

$$y \in X,\ u \in U,\ v \in \mathbb{R}_+^q,\ w \in \mathbb{R}^r,\ \alpha^i \in \mathbb{R}^{\ell_i},\ \beta^i \in \mathbb{R}^{m_i},\ i \in \underline{p},\ \gamma^j \in \mathbb{R}^{n_j},\ j \in \underline{q}, \tag{3.13}$$

where for each $i \in \underline{p}$, $N_i^{\circ}(y, \alpha) = f_i(y) + \langle \alpha^i, A_i y\rangle$ and $D_i^{\circ}(y, \beta) = g_i(y) - \langle \beta^i, B_i y\rangle$;

$$(\tilde{\text{D}}\text{I}) \quad \text{Maximize} \left(\frac{f_1(y) + \langle \alpha^1, A_1 y\rangle}{g_1(y) - \langle \beta^1, B_1 y\rangle}, \ldots, \frac{f_p(y) + \langle \alpha^p, A_p y\rangle}{g_p(y) - \langle \beta^p, B_p y\rangle} \right)$$

subject to (3.10)–(3.13) and

$$\Big\langle \sum_{i=1}^{p} u_i\{D_i^\circ(y,\beta)[\nabla f_i(y)+A_i^T\alpha^i]-N_i^\circ(y,\alpha)[\nabla g_i(y)-B_i^T\beta^i]\} + \sum_{j=1}^{q} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k=1}^{r} w_k\nabla H_k(y), \eta(x,y)\Big\rangle \geqq 0 \quad \text{for all } x\in\mathbb{F}, \tag{3.14}$$

where η is a function from $X\times X$ to $\mathbb{R}^n$.

The structures of the first two problems designated above as (CI) and ($\tilde{\text{C}}$I), which can be proved under appropriate (η,ρ)-invexity hypotheses to be dual problems for (P), are based directly on the form and contents of the necessary efficiency conditions of Theorem 2.1. This is, of course, the standard method for constructing Wolfe-type dual problems. However, a careful examination of the form and features of (CI) and ($\tilde{\text{C}}$I) (as well as the proofs of the weak and strong duality theorems for (P)–(DI) given below), will readily reveal the fact that the constraints (3.5) and (3.6) are essentially superfluous and their omission will not invalidate the duality relations between (P) and (CI), and (P) and ($\tilde{\text{C}}$I). More specifically, if (3.5) and (3.6) are deleted and the remaining constraints of (CI) and ($\tilde{\text{C}}$I) are modified accordingly, then one obtains the reduced versions (DI) and ($\tilde{\text{D}}$I).

Comparing (DI) and ($\tilde{\text{D}}$I), we see that ($\tilde{\text{D}}$I) is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for ($\tilde{\text{D}}$I), but the converse is not necessarily true. Furthermore, we observe that (3.9) is a system of n equations, whereas (3.14) is a single inequality. Clearly, from a computational point of view, (DI) is preferable to ($\tilde{\text{D}}$I) because of the dependence of (3.14) on the feasible set of (P).

Despite these apparent differences, however, it turns out that the statements and proofs of all the duality theorems for (P)–(DI) and (P)–($\tilde{D}$I) are almost identical and, therefore, we shall consider only the pair (P)–(DI). Similarly, it is easily seen that all of the duality theorems established for (P)–(DI) can readily be altered and restated for (P)–(CI) and (P)–($\tilde{\text{C}}$I).

For the sake of economy of space and expression, we shall use the following list of symbols in the statements and proofs of our duality theorems:

$$\mathcal{A}_i(x,\alpha)=f_i(x)+\langle\alpha^i, A_ix\rangle,\;\; i\in\underline{p},$$
$$\mathcal{B}_i(x,\beta)=-g_i(x)+\langle\beta^i, B_ix\rangle,\;\; i\in\underline{p},$$
$$\mathcal{C}_j(x,\gamma)=G_j(x)+\langle\gamma^j, C_jx\rangle,\;\; j\in\underline{q},$$
$$\mathcal{D}_k(x,w)=w_kH_k(x),\;\; k\in\underline{r},$$

$$\begin{aligned}
\mathcal{E}_i(x,y,\alpha,\beta) &= D_i^\circ(y,\beta)[f_i(x)+\langle\alpha^i,A_ix\rangle]-N_i^\circ(y,\alpha)[g_i(x)\\
&\quad -\langle\beta^i,B_ix\rangle],\ \ i\in\underline{p},\\
\mathcal{C}(x,v,\gamma) &= \sum_{j=1}^{q} v_j[G_j(x)+\langle\gamma^j,C_jx\rangle],\\
\mathcal{D}(x,w) &= \sum_{k=1}^{r} w_kH_k(x),\\
\mathcal{E}(x,y,u,\alpha,\beta) &= \sum_{i=1}^{p} u_i\{D_i^\circ(y,\beta)[f_i(x)+\langle\alpha^i,A_ix\rangle]-N_i^\circ(y,\alpha)[g_i(x)\\
&\quad -\langle\beta^i,B_ix\rangle]\},\\
\mathcal{F}(x,v,w,\gamma) &= \sum_{j=1}^{q} v_j[G_j(x)+\langle\gamma^j,C_jx\rangle]+\sum_{k=1}^{r} w_kH_k(x),\\
J_+(v) &= \{j\in\underline{q}: v_j>0\}\ \text{for fixed}\ v\in\mathbb{R}^q_+,\\
K_*(w) &= \{k\in\underline{r}: w_k\neq 0\}\ \text{for fixed}\ w\in\mathbb{R}^r,\\
\alpha &= (\alpha^1,\alpha^2,\ldots,\alpha^p),\\
\beta &= (\beta^1,\beta^2,\ldots,\beta^p),\\
\gamma &= (\gamma^1,\gamma^2,\ldots,\gamma^q).
\end{aligned}$$

In the sequel, we shall make frequent use of the well-known generalized Cauchy inequality which is formally stated in the following lemma.

LEMMA 3.1 [7]. *For each* $a,b\in\mathbb{R}^m$, $a^Tb\leqq\|a\|^*\|b\|$.

Throughout this paper, we assume that $N_i^\circ(y,\alpha)\geqq 0$, $D_i^\circ(y,\beta)>0$, $i\in\underline{p}$, for all y, α, and β such that $(y,u,v,w,\alpha,\beta,\gamma)$ is a feasible solution of the dual problem under consideration.

The next two theorems show that (DI) is a dual problem for (P).

THEOREM 3.1 (Weak Duality). *Let* x *and* $z\equiv(y,u,v,w,\alpha,\beta,\gamma)$ *be arbitrary feasible solutions of* (P) *and* (*DI*), *respectively, and assume that either one of the following two sets of hypotheses is satisfied:*

(a) (i) *for each* $i\in\underline{p}$, $\mathcal{A}_i(\cdot,\alpha)$ *is* $(\eta,\bar\rho_i)$*-invex and* $\mathcal{B}_i(\cdot,\beta)$ *is* $(\eta,\tilde\rho_i)$*-invex at* y;
(ii) *for each* $j\in J_+\equiv J_+(v)$, $\mathcal{C}_j(\cdot,\gamma)$ *is* $(\eta,\hat\rho_j)$*-invex at* y;
(iii) *for each* $k\in K_*\equiv K_*(w)$, w_kH_k *is* $(\eta,\breve\rho_k)$*-invex at* y;
(iv) $\sum_{i=1}^{p}u_i[D_i^\circ(y,\beta)\bar\rho_i+N_i^\circ(y,\alpha)\tilde\rho_i]+\sum_{j\in J_+}v_j\hat\rho_j+\sum_{k=1}^{r}\breve\rho_k\geqq 0$;

(b) *The Lagrangian-type function* $L(\cdot, y, u, v, w, \alpha, \beta, \gamma): X \to \mathbb{R}$ *defined by*

$$L(x, y, u, v, w, \alpha, \beta, \gamma) = \sum_{i=1}^{p} u_i\{D_i^\circ(y, \beta)[f_i(x) + \langle\alpha^i, A_i x\rangle] \\ -N_i^\circ(y, \alpha)[g_i(x) - \langle\beta^i, B_i x\rangle]\} \\ + \sum_{j=1}^{q} v_j[G_j(x) + \langle\gamma^j, C_j x\rangle] + \sum_{k=1}^{r} w_k H_k(x)$$

is $(\eta, 0)$*-pseudoinvex at* y.

Then $\varphi(x) \nleq \psi(z)$*, where* $\psi = (\psi_1, \ldots, \psi_p)$ *is the objective function of* (*DI*).

Proof. (a) Keeping in mind that0 $u > 0$, $v \geqq 0$, $N_i^\circ(y, \alpha) \geqq 0$, and $D_i^\circ(y, \beta) > 0$, $i \in \underline{p}$, we have

$$\sum_{i=1}^{p} u_i\{D_i^\circ(y, \beta)[f_i(x) + \|A_i x\|_{a(i)}] - N_i^\circ(y, \alpha)[g_i(x) - \|B_i x\|_{b(i)}]\}$$

$$= \sum_{i=1}^{p} u_i\{D_i^\circ(y, \beta)\{f_i(x) + \|A_i x\|_{a(i)} - [f_i(y) - \langle\alpha^i, A_i y\rangle]\}$$
$$-N_i^\circ(y, \alpha)\{g_i(x) - \|B_i x\|_{b(i)} - [g_i(y) - \langle\beta^i, B_i y\rangle]\}\}$$

(by the definitions of $N_i^\circ(y, \alpha)$ and $D_i^\circ(y, \beta)$, $i \in \underline{p}$)

$$\geqq \sum_{i=1}^{p} u_i\{D_i^\circ(y, \beta)[f_i(x) + \|\alpha^i\|_{a(i)}^* \|A_i x\|_{a(i)}]$$
$$-N_i^\circ(y, \alpha)[g_i(x) - \|\beta^i\|_{b(i)}^* \|B_i x\|_{b(i)}] - D_i^\circ(y, \beta)[f_i(y) + \langle\alpha^i, A_i y\rangle]$$
$$+N_i^\circ(y, \alpha)[g_i(y) - \langle\beta^i, B_i y\rangle]\} \text{ (by (3.11))}$$

$$\geqq \sum_{i=1}^{p} u_i\{D_i^\circ(y, \beta)\{f_i(x) + \langle\alpha^i, A_i x\rangle - [f_i(y) + \langle\alpha^i, A_i y\rangle]\}$$
$$-N_i^\circ(y, \alpha)\{g_i(x) - \langle\beta^i, B_i x\rangle - [g_i(y) - \langle\beta^i, B_i y\rangle]\}\}$$

(by Lemma 3.1)

$$\geqq \sum_{i=1}^{p} u_i\{\langle D_i^\circ(y, \beta)[\nabla f_i(y) + A_i^T \alpha^i]$$
$$-N_i^\circ(y, \alpha)[\nabla g_i(y) - B_i^T \beta^i], \eta(x, y)\rangle + [D_i^\circ(y, \beta)\bar{\rho}_i$$
$$+N_i^\circ(y, \alpha)\tilde{\rho}_i]\|x - y\|^2\} \quad \text{(by (i))}$$

$$
\begin{aligned}
&= -\Big\langle \sum_{j=1}^{q} v_j[\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x, y)\Big\rangle \\
&\quad + \sum_{i=1}^{p} u_i[D_i^\circ(y,\beta)\bar{\rho}_i + N_i^\circ(y,\alpha)\tilde{\rho}_i]\|x - y\|^2 \quad \text{(by (3.9))} \\
&\geqq \sum_{j=1}^{q} v_j\{G_j(y) + \langle \gamma^j, C_j y\rangle - [G_j(x) + \langle \gamma^j, C_j x\rangle]\} + \sum_{k=1}^{r} w_k H_k(y) \\
&\quad + \Big(\sum_{i=1}^{p} u_i[D_i^\circ(y,\beta)\bar{\rho}_i + N_i^\circ(y,\alpha)\tilde{\rho}_i] + \sum_{j\in J_+} v_j\hat{\rho}_j + \sum_{k=1}^{r}\breve{\rho}_k\Big)\|x - y\|^2 \\
&\quad \text{(by (ii), (iii), and primal feasibility of } x) \\
&\geqq -\sum_{j=1}^{q} v_j[G_j(x) + \|\gamma^j\|_{c(j)}^{*}\|C_j x\|_{c(j)}] + \sum_{j=1}^{q} v_j[G_j(y) + \langle \gamma^j, C_j y\rangle] \\
&\quad + \sum_{k=1}^{r} w_k H_k(y) \text{ (by (iv) and Lemma 3.1)} \\
&\geqq -\sum_{j=1}^{q} v_j[G_j(x) + \|C_j x\|_{c(j)}] + \sum_{j=1}^{q} v_j[G_j(y) + \langle \gamma^j, C_j y\rangle] \\
&\quad + \sum_{k=1}^{r} w_k H_k(y) \text{ (by (3.12))} \\
&\geqq \sum_{j=1}^{q} v_j[G_j(y) + \langle \gamma^j, C_j y\rangle] + \sum_{k=1}^{r} w_k H_k(y) \\
&\quad \text{(by the primal feasibility of } x).
\end{aligned}
$$

In view of (3.10), the above inequality reduces to

$$
\sum_{i=1}^{p} u_i\{D_i^\circ(y,\beta)[f_i(x) + \|A_i x\|_{a(i)}] - N_i^\circ(y,\alpha)[g_i(x) - \|B_i x\|_{b(i)}]\} \geqq 0. \tag{3.15}
$$

Since $u > 0$, (3.15) implies that

$$
\begin{aligned}
&\Big(D_1^\circ(y,\beta)[f_1(x) + \|A_1 x\|_{a(1)}] - N_1^\circ(y,\alpha)[g_1(x) - \|B_1 x\|_{b(1)}], \ldots, \\
&\quad D_p^\circ(y,\beta)[f_p(x) + \|A_p x\|_{a(p)}] - N_p^\circ(y,\alpha)[g_p(x) - \|B_p x\|_{b(p)}]\Big) \\
&\quad \nleq (0, \ldots, 0),
\end{aligned}
$$

which, in turn, implies that

$$\varphi(x)=\left(\frac{f_1(x)+\|A_1x\|_{a(1)}}{g_1(x)-\|B_1x\|_{b(1)}},\ldots,\frac{f_p(x)+\|A_px\|_{a(p)}}{g_p(x)-\|B_px\|_{b(p)}}\right)$$
$$\nleq\left(\frac{f_1(y)+\langle\alpha^1,A_1y\rangle}{g_1(y)-\langle\beta^1,B_1y\rangle},\ldots,\frac{f_p(y)+\langle\alpha^p,A_py\rangle}{g_p(y)-\langle\beta^p,B_py\rangle}\right)=\psi(z).$$

(b) From our $(\eta, 0)$-pseudoinvexity assumption and (3.9) it follows that $L(x,y,u,v,w,\alpha,\beta,\gamma)\geqq L(y,y,u,v,w,\alpha,\beta,\gamma)$. In view of (3.10) and primal feasibility of x, this inequality reduces to

$$\sum_{i=1}^{p}u_i\{D_i^{\circ}(y,\beta)[f_i(x)+\langle\alpha^i,A_ix\rangle]-N_i^{\circ}(y,\alpha)[g_i(x)-\langle\beta^i,B_ix\rangle]\}$$
$$+\sum_{j=1}^{q}v_j[G_j(x)+\langle\gamma^j,C_jx\rangle]\geqq 0.$$

Using this inequality and bearing in mind that $u>0$, $v\geqq 0$, $N_i^{\circ}(y,\alpha)\geqq 0$, and $D_i^{\circ}(y,\beta)>0$, $i\in\underline{p}$, we see that

$$0\leqq\sum_{i=1}^{p}u_i\{D_i^{\circ}(y,\beta)[f_i(x)+\|\alpha^i\|_{a(i)}^{*}\|A_ix\|_{a(i)}]-N_i^{\circ}(y,\alpha)[g_i(x)$$
$$-\|\beta^i\|_{b(i)}^{*}\|B_ix\|_{b(i)}]\}+\sum_{j=1}^{q}v_j[G_j(x)$$
$$+\|\gamma^j\|_{c(j)}^{*}\|C_jx\|_{c(j)}]\ \text{(by Lemma 3.1)}$$
$$\leqq\sum_{i=1}^{p}u_i\{D_i^{\circ}(y,\beta)[f_i(x)+\|A_ix\|_{a(i)}]-N_i^{\circ}(y,\alpha)[g_i(x)-\|B_ix\|_{b(i)}]\}$$
$$+\sum_{j=1}^{q}v_j[G_j(x)+\|C_jx\|_{c(j)}]\ \text{(by (3.11) and (3.12))}$$
$$\leqq\sum_{i=1}^{p}u_i\{D_i^{\circ}(y,\beta)[f_i(x)+\|A_ix\|_{a(i)}]-N_i^{\circ}(y,\alpha)[g_i(x)-\|B_ix\|_{b(i)}]\}$$
$$\text{(by the primal feasibility of } x\text{)},$$

which is (3.15), and hence the rest of the proof is identical to that of part (a). □

THEOREM 3.2 (Strong Duality). *Let x^* be a normal efficient solution of (P) and assume that either one of the two sets of conditions specified in*

Theorem 3.1 is satisfied for all feasible solutions of (*DI*). *Then there exist* $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}, i \in \underline{p}$, *and* $\gamma^{*j} \in \mathbb{R}^{n_j}, j \in \underline{q}$, *such that* $z^* \equiv (x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is an efficient solution of* (*DI*) *and* $\varphi(x^*) = \psi(z^*)$.

Proof. Since x^* is a normal efficient solution of (P), by Theorem 2.1, there exist u^*, v^*, w^*, α^{*i}, $\beta^{*i}, i \in \underline{p}$, and $\gamma^{*j}, j \in \underline{q}$, as specified above, such that z^* is a feasible solution of (*DI*). If it were not efficient, then there would exist a feasible solution $\hat{z} \equiv (\hat{x}, \hat{u}, \hat{v}, \hat{w}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ of (*DI*) such that $\psi(\hat{z}) \geqslant \psi(z^*)$. But $\psi(z^*) = \varphi(x^*)$ and hence $\psi(\hat{z}) \geqslant \varphi(x^*)$, which contradicts Theorem 3.1. Therefore, we conclude that z^* is an efficient solution of (*DI*). □

We also have the following converse duality result for (P)–(DI).

THEOREM 3.3 (Strict Converse Duality). *Let* x^* *be an efficient solution of* (*P*), *let* $\tilde{z} \equiv (\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ *be a feasible solution of* (*DI*) *such that*

$$\sum_{i=1}^{p} \tilde{u}_i \{D_i^\circ(\tilde{x}, \tilde{\beta})[f_i(x^*) + \|A_i x^*\|_{a(i)}] - N_i^\circ(\tilde{x}, \tilde{\alpha})[g_i(x^*) - \|B_i x^*\|_{b(i)}]\} \leqq 0. \tag{3.16}$$

Furthermore, assume that either one of the following two sets of hypotheses is satisfied:

(a) *The assumptions of part* (a) *of Theorem* 3.1 *are satisfied for the feasible solution* $\tilde{z}$ *of* (*DI*) *and* $\mathcal{A}_i(\cdot, \tilde{\alpha})$ *is strictly* $(\eta, \bar{\rho}_i)$*-invex at* $\tilde{x}$ *for at least one index* $i \in \underline{p}$, *or* $\mathcal{B}_i(\cdot, \tilde{\beta})$ *is strictly* $(\eta, \tilde{\rho}_i)$*-invex at* $\tilde{x}$ *for at least one index* $i \in \underline{p}$, *or* $\mathcal{C}_j(\cdot, \gamma)$ *is strictly* $(\eta, \hat{\rho}_j)$*-invex at* $\tilde{x}$ *for at least one index* $j \in \underline{q}$ *with the corresponding component* $\tilde{v}_j$ *of* $\tilde{v}$ *positive, or* $\tilde{w}_k H_k$ *is strictly* $(\eta, \breve{\rho}_k)$*-invex at* $\tilde{x}$ *for at least one index* $k \in K_*(\tilde{x})$, *or*

$$\sum_{i=1}^{p} \tilde{u}_i [D_i^\circ(\tilde{x}, \tilde{\beta})\bar{\rho}_i + N_i^\circ(\tilde{x}, \tilde{\alpha})\tilde{\rho}_i] + \sum_{j \in J_+} \tilde{v}_j \hat{\rho}_j + \sum_{k=1}^{r} \breve{\rho}_k > 0.$$

(b) *The assumptions of part* (b) *of Theorem* 3.1 *are satisfied for the feasible solution* $\tilde{z}$ *of* (*DI*), *and the function* $L(\cdot, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ *is strictly* $(\eta, 0)$*-pseudoinvex at* $\tilde{x}$.

Then $\tilde{x} = x^*$, *that is,* $\tilde{x}$ *is an efficient solution of* (*P*), *and* $\varphi(x^*) = \psi(\tilde{z})$.

Proof. (a) Suppose to the contrary that $\tilde{x} \neq x^*$. Now proceeding as in the proof of Theorem 3.1 (with x replaced by x^* and z by $\tilde{z}$) and using any of the conditions set forth above, we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i \{D_i^\circ(\tilde{x}, \tilde{\beta})[f_i(x^*) + \|A_i x^*\|_{a(i)}] - N_i^\circ(\tilde{x}, \tilde{\alpha})[g_i(x^*) - \|B_i x^*\|_{b(i)}]\} > 0,$$

which contradicts (3.16). Therefore, we conclude that $\tilde{x}=x^*$ and $\varphi(x^*)=\psi(\tilde{z})$.

(b) The proof is similar to that of part (a). □

4. Duality Model II

In this section, we consider certain variants of (CI), ($\tilde{\text{C}}$I), (DI), and ($\tilde{\text{D}}$I) that allow for a greater variety of generalized (η,ρ)-invexity conditions under which duality can be established. These duality models have the following forms:

$$\text{(CII)}\quad \text{Maximize}\left(\frac{f_1(y)+\|A_1y\|_{a(1)}}{g_1(y)-\|B_1y\|_{b(1)}},\dots,\frac{f_p(y)+\|A_py\|_{a(p)}}{g_p(y)-\|B_py\|_{b(p)}}\right)$$

subject to (3.1), (3.3)–(3.7), and

$$v_j[G_j(y)+\|C_jy\|_{c(j)}]\geqq 0,\quad j\in\underline{q},\tag{4.1}$$

$$w_kH_k(y)\geqq 0,\quad k\in\underline{r};\tag{4.2}$$

$$(\tilde{\text{C}}\text{II})\quad \text{Maximize}\left(\frac{f_1(y)+\|A_1y\|_{a(1)}}{g_1(y)-\|B_1y\|_{b(1)}},\dots,\frac{f_p(y)+\|A_py\|_{a(p)}}{g_p(y)-\|B_py\|_{b(p)}}\right)$$

subject to (3.3)–(3.8), (4.1), and (4.2);

$$\text{(DII)}\quad \text{Maximize}\left(\frac{f_1(y)+\langle\alpha^1,A_1y\rangle}{g_1(y)-\langle\beta^1,B_1y\rangle},\dots,\frac{f_p(y)+\langle\alpha^p,A_py\rangle}{g_p(y)-\langle\beta^p,B_py\rangle}\right)$$

subject to

$$\sum_{i=1}^{p}u_i\{D_i^\circ(y,\beta)[\nabla f_i(y)+A_i^T\alpha^i]-N_i^\circ(y,\alpha)[\nabla g_i(y)-B_i^T\beta^i]\}$$
$$+\sum_{j=1}^{q}v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k=1}^{r}w_k\nabla H_k(y)=0,\tag{4.3}$$

$$v_j[G_j(y)+\langle\gamma^j,C_jy\rangle]\geqq 0,\quad j\in\underline{q},\tag{4.4}$$

$$w_kH_k(y)\geqq 0,\quad k\in\underline{r},\tag{4.5}$$

$$\|\alpha^i\|^*_{a(i)}\leqq 1,\quad \|\beta^i\|^*_{b(i)}\leqq 1,\quad i\in\underline{p},\tag{4.6}$$

$$\|\gamma^j\|^*_{c(j)} \leqq 1, \quad j \in \underline{q}, \tag{4.7}$$

$$y \in X,\ u \in U,\ v \in \mathbb{R}^q_+,\ w \in \mathbb{R}^r,\ \alpha^i \in \mathbb{R}^{\ell_i},\ \beta^i \in \mathbb{R}^{m_i},\ i \in \underline{p}, \gamma^j \in \mathbb{R}^{n_j},\ j \in \underline{q}; \tag{4.8}$$

$$(\tilde{\text{D}}\text{II}) \quad \text{Maximize} \left(\frac{f_1(y) + \langle \alpha^1, A_1 y \rangle}{g_1(y) - \langle \beta^1, B_1 y \rangle}, \ldots, \frac{f_p(y) + \langle \alpha^p, A_p y \rangle}{g_p(y) - \langle \beta^p, B_p y \rangle} \right)$$

subject to (3.14) and (4.4)–(4.8).

The remarks and observations made earlier about the relationships among (CI), ($\tilde{\text{C}}$I), (DI), and ($\tilde{\text{D}}$I) are, of course, also valid for (CII), ($\tilde{\text{C}}$II), (DII), and ($\tilde{\text{D}}$II). As in the preceding section, we shall work with the reduced versions (DII) and ($\tilde{\text{D}}$II), and, in particular, consider the pair (P)–(DII).

As will be demonstrated throughout this section, duality for (P)–(DII) can be proved under a great variety of generalized (η, ρ)-invexity hypotheses. Our first collection of weak duality results is given in the next theorem in which separate (η, ρ)-invexity conditions are imposed on the functions $\mathcal{A}_i(\cdot, \alpha)$ and $\mathcal{B}_i(\cdot, \beta)$, $i \in \underline{p}$.

THEOREM 4.1 (Weak Duality). *Let x and $z \equiv (y, u, v, w, \alpha, \beta, \gamma)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha)$ is $(\eta, \bar{\rho}_i)$-invex and $\mathcal{B}_i(\cdot, \beta)$ is $(\eta, \tilde{\rho}_i)$-invex at y;*
(ii) *for each $j \in J_+ \equiv J_+(v)$, $\mathcal{C}_j(\cdot, \gamma)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at y;*
(iii) *for each $k \in K_* \equiv K_*(w)$, $\mathcal{D}_k(\cdot, w)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at y;*
(iv) *$\rho^* + \sum_{j \in J_+} v_j \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$, where $\rho^* = \sum_{i=1}^p u_i [D_i^\circ(y, \beta)\bar{\rho}_i + N_i^\circ(y, \alpha)\tilde{\rho}_i]$;*

(b) (i) *for each $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha)$ is $(\eta, \bar{\rho}_i)$-invex and $\mathcal{B}_i(\cdot, \beta)$ is $(\eta, \tilde{\rho}_i)$-invex at y;*
(ii) *$\mathcal{C}(\cdot, v, \gamma)$ is $(\eta, \hat{\rho})$-quasiinvex at y;*
(iii) *for each $k \in K_*$, $\mathcal{D}_k(\cdot, w)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at y;*
(iv) *$\rho^* + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$;*

(c) (i) *for each $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha)$ is $(\eta, \bar{\rho}_i)$-invex and $\mathcal{B}_i(\cdot, \beta)$ is $(\eta, \tilde{\rho}_i)$-invex at y;*
(ii) *for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at y;*
(iii) *$\mathcal{D}(\cdot, w)$ is $(\eta, \breve{\rho})$-quasiinvex at y;*
(iv) *$\rho^* + \sum_{j \in J_+} v_j \hat{\rho}_j + \breve{\rho} \geqq 0$;*

(d) (i) *for each $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha)$ is $(\eta, \bar{\rho}_i)$-invex and $\mathcal{B}_i(\cdot, \beta)$ is $(\eta, \tilde{\rho}_i)$-invex at y;*
(ii) *$\mathcal{C}(\cdot, v, \gamma)$ is $(\eta, \hat{\rho})$-quasiinvex at y;*

(iii) $\mathcal{D}(\cdot, w)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* y*;*
(iv) $\rho^* + \hat{\rho} + \breve{\rho} \geqq 0$*;*

(e) (i) *for each* $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha)$ *is* $(\eta, \bar{\rho}_i)$*-invex and* $\mathcal{B}_i(\cdot, \beta)$ *is* $(\eta, \tilde{\rho}_i)$*-invex at* y*;*
(ii) $\mathcal{F}(\cdot, v, w, \gamma)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* y*;*
(iii) $\rho^* + \hat{\rho} \geqq 0$*;*

Then $\varphi(x) \nleq \theta(z)$*, where* $\theta = (\theta_1, \ldots, \theta_p)$ *is the objective function of* (*DII*).

Proof. (a) Since for each $j \in J_+$,

$$\begin{aligned} G_j(x) + \langle \gamma^j, C_j x \rangle &\leqq G_j(x) + \|\gamma^j\|^*_{c(j)} \|C_j x\|_{c(j)} \text{ (by Lemma 3.1)} \\ &\leqq G_j(x) + \|C_j x\|_{c(j)} \text{ (by (4.7))} \\ &\leqq 0 \text{ (since } x \in \mathbb{F}) \\ &\leqq G_j(y) + \langle \gamma^j, C_j y \rangle; \text{ (by (4.4))}, \end{aligned}$$

in view of (ii) we have

$$\langle \nabla G_j(y) + C_j^T \gamma^j, \eta(x, y) \rangle \leqq -\hat{\rho}_j \|x - y\|^2.$$

As $v_j \geqq 0$ for each $j \in \underline{q}$, and $v_j = 0$ for each $j \in \underline{q} \backslash J_+$ (complement of J_+ relative to $\underline{q}$), the above inequalities yield

$$\left\langle \sum_{j=1}^{q} v_j [\nabla G_j(y) + C_j^T \gamma^j], \eta(x, y) \right\rangle \leqq -\sum_{j \in J_+} v_j \hat{\rho}_j \|x - y\|^2. \tag{4.9}$$

In a similar manner we can show that (iii) leads to the following inequality:

$$\left\langle \sum_{k=1}^{r} \nabla w_k H_k(y), \eta(x, y) \right\rangle \leqq -\sum_{k \in K_*} \breve{\rho}_k \|x - y\|^2. \tag{4.10}$$

Bearing in mind that $u > 0$, $v \geqq 0$, $N_i^\circ(y, \alpha) \geqq 0$, and $D_i^\circ(y, \beta) > 0$, $i \in \underline{p}$, we have

$$\begin{aligned} &\sum_{i=1}^{p} u_i \{D_i^\circ(y, \beta)[f_i(x) + \|A_i x\|_{a(i)}] - N_i^\circ(y, \alpha)[g_i(x) - \|B_i x\|_{b(i)}]\} \\ &\quad \geqq \sum_{i=1}^{p} u_i \{D_i^\circ(y, \beta)[f_i(x) + \|\alpha^i\|^*_{a(i)} \|A_i x\|_{a(i)}] \\ &\qquad - N_i^\circ(y, \alpha)[g_i(x) - \|\beta^i\|^*_{b(i)} \|B_i x\|_{b(i)}]\} \text{ (by (4.6))} \\ &\quad \geqq \sum_{i=1}^{p} u_i \{D_i^\circ(y, \beta)[f_i(x) + \langle \alpha^i, A_i x \rangle] \\ &\qquad - N_i^\circ(y, \alpha)[g_i(x) - \langle \beta_i^i, B_i x \rangle]\} \text{ (by Lemma 3.1)} \end{aligned}$$

$$= \sum_{i=1}^{p} u_i \{ D_i^{\circ}(y,\beta)\{f_i(x) + \langle \alpha^i, A_i x\rangle - [f_i(y) + \langle \alpha^i, A_i y\rangle]\}$$
$$-N_i^{\circ}(y,\alpha)\{g_i(x) - \langle \beta^i, B_i x\rangle - [g_i(y) - \langle \beta^i, B_i y\rangle]\}\}$$
(by the definitions of $N_i^{\circ}(y,\alpha)$ and $D_i^{\circ}(y,\beta)$, $i \in \underline{p}$)
$$\geqq \sum_{i=1}^{p} u_i \{\langle D_i^{\circ}(y,\beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^{\circ}(y,\alpha)[\nabla g_i(y) - B_i^T \beta^i], \eta(x,y)\rangle$$
$$+[D_i^{\circ}(y,\beta)\bar{\rho}_i + N_i^{\circ}(y,\alpha)\tilde{\rho}_i]\|x-y\|^2\} \text{ (by (i))}$$
$$= -\Big\langle \sum_{j=1}^{q} v_j[\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x,y)\Big\rangle$$
$$+ \sum_{i=1}^{p} u_i [D_i^{\circ}(y,\beta)\bar{\rho}_i + N_i^{\circ}(y,\alpha)\tilde{\rho}_i)]\|x-y\|^2 \text{ (by (4.3))}$$
$$\geqq \Big(\rho^* + \sum_{j\in J_+} v_j \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k\Big)\|x-y\|^2 \text{ (by (4.9) and (4.10))}$$
$$\geqq 0 \text{ (by (iv))}.$$

As shown in the proof of Theorem 3.1, this inequality leads to the desired conclusion that $\varphi(x) \nleq \theta(z)$.

(b) As shown in part (a), for each $j \in J_+$, we have $G_j(x) + \langle \gamma^j, C_j x\rangle \leqq G_j(y) + \langle \gamma^j, C_j y\rangle$ and hence

$$\sum_{j=1}^{q} v_j [G_j(x) + \langle \gamma^j, C_j x\rangle] \leqq \sum_{j=1}^{q} v_j [G_j(y) + \langle \gamma^j, C_j y\rangle],$$

which in view of (ii) implies that

$$\Big\langle \sum_{j=1}^{q} v_j [\nabla G_j(y) + C_j^T \gamma^j], \eta(x,y) \Big\rangle \leqq -\hat{\rho}\|x-y\|^2.$$

Now proceeding as in the proof of part (a) and using this inequality instead of (4.9), we arrive at the conclusion that $\varphi(x) \nleq \theta(z)$.

(c)–(e): The proofs are similar to those of parts (a) and (b). □

THEOREM 4.2 (Strong Duality). *Let x^* be a normal efficient solution of (P) and assume that any one of the five sets of conditions set forth in Theorem 4.1 is satisfied for all feasible solutions of (DII). Then there exist $u^* \in U$, $v^* \in \mathbb{R}_+^q$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}, i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}, j \in \underline{q}$, such that $z^* \equiv (x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is an efficient solution of (DII) and $\varphi(x^*) = \theta(z^*)$.*

Proof. The proof is similar to that of Theorem 3.2. □

THEOREM 4.3 (Strict Converse Duality). *Let x^* be an efficient solution of (P) and let $\tilde{z} \equiv (\tilde{x}, \tilde{\lambda}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ be a feasible solution of (DII) such that*

$$\sum_{i=1}^{p} \tilde{u}_i \{D_i^\circ(\tilde{x}, \tilde{\beta})[f_i(x) + \|A_i x\|_{a(i)}] - N_i^\circ(\tilde{x}, \tilde{\alpha})[g_i(x) - \|B_i x\|_{b(i)}]\} \leqq 0. \tag{4.11}$$

Furthermore, assume that any one of the following five sets of conditions is satisfied:

(a) *The assumptions specified in part* (a) *of Theorem* 4.1 *are satisfied for the feasible solution $\tilde{z}$ of (DII) and $\mathcal{A}_i(\cdot, \tilde{\alpha})$ is strictly $(\eta, \bar{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{B}_i(\cdot, \tilde{\beta})$ is strictly $(\eta, \tilde{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{C}_j(\cdot, \tilde{\gamma})$ is strictly $(\eta, \hat{\rho}_j)$-pseudoinvex at $\tilde{x}$ for at least one $j \in J_+(\tilde{v}_j)$, or $\mathcal{D}_k(\cdot, \tilde{w})$ is strictly $(\eta, \breve{\rho}_k)$-pseudoinvex at $\tilde{x}$ for at least one $k \in K_*(\tilde{w})$, or $\rho^* + \sum_{j \in J_+} \tilde{v}_j \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k > 0$, where $\rho^* = \sum_{i=1}^{p} \tilde{u}_i [D_i^\circ(\tilde{x}, \tilde{\beta})\bar{\rho}_i + N_i^\circ(\tilde{x}, \tilde{\alpha})\tilde{\rho}_i]$.*

(b) *The assumptions specified in part* (b) *of Theorem* 4.1 *are satisfied for the feasible solution $\tilde{z}$ of (DII) and $\mathcal{A}_i(\cdot, \tilde{\alpha})$ is strictly $(\eta, \bar{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{B}_i(\cdot, \tilde{\beta})$ is strictly $(\eta, \tilde{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{C}(\cdot, \tilde{v}, \tilde{\gamma})$ is strictly $(\eta, \hat{\rho})$-pseudoinvex at $\tilde{x}$, or $\mathcal{D}_k(\cdot, \tilde{w})$ is strictly $(\eta, \breve{\rho}_k)$-pseudoinvex at $\tilde{x}$ for at least one $k \in K_*(\tilde{w})$, or $\rho^* + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k > 0$.*

(c) *The assumptions specified in part* (c) *of Theorem* 4.1 *are satisfied for the feasible solution $\tilde{z}$ of (DII) and $\mathcal{A}_i(\cdot, \tilde{\alpha})$ is strictly $(\eta, \bar{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{B}_i(\cdot, \tilde{\beta})$ is strictly $(\eta, \tilde{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{C}_j(\cdot, \tilde{\gamma})$ is strictly $(\eta, \hat{\rho}_j)$-pseudoinvex at $\tilde{x}$ for at least one $j \in J_+(\tilde{v}_j)$, or $\mathcal{D}(\cdot, \tilde{w})$ is strictly $(\eta, \breve{\rho})$-pseudoinvex at $\tilde{x}$, or $\rho^* + \sum_{j \in J_+} \tilde{v}_j \hat{\rho}_j + \breve{\rho} > 0$.*

(d) *The assumptions specified in part* (d) *of Theorem* 4.1 *are satisfied for the feasible solution $\tilde{z}$ of (DII) and $\mathcal{A}_i(\cdot, \tilde{\alpha})$ is strictly $(\eta, \bar{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{B}_i(\cdot, \tilde{\beta})$ is strictly $(\eta, \tilde{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{C}(\cdot, \tilde{v}, \tilde{\gamma})$ is strictly $(\eta, \hat{\rho})$-pseudoinvex at $\tilde{x}$, or $\mathcal{D}(\cdot, \tilde{w})$ is strictly $(\eta, \breve{\rho})$-pseudoinvex at $\tilde{x}$, or $\rho^* + \hat{\rho} + \breve{\rho} > 0$.*

(e) *The assumptions specified in part* (e) *of Theorem* 4.1 *are satisfied for the feasible solution $\tilde{z}$ of (DII) and $\mathcal{A}_i(\cdot, \tilde{\alpha})$ is strictly $(\eta, \bar{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{B}_i(\cdot, \tilde{\beta})$ is strictly $(\eta, \tilde{\rho}_i)$-invex at $\tilde{x}$ for at least one $i \in \underline{p}$, or $\mathcal{F}(\cdot, \tilde{v}, \tilde{w}, \tilde{\gamma})$ is strictly $(\eta, \breve{\rho})$-pseudoinvex at $\tilde{x}$, or $\rho^* + \hat{\rho} > 0$.*

Then $\tilde{x} = x^*$ and $\varphi(x^*) = \theta(\tilde{x})$.

Proof. The proof is similar to that of Theorem 3.3. □

In Theorem 4.1, separate (η, ρ)-invexity assumptions were imposed on the functions $\mathcal{A}_i(\cdot, \alpha)$ and $\mathcal{B}_i(\cdot, \beta)$, $i \in \underline{p}$. In the remainder of this section, we shall formulate several duality results in which various generalized (η, ρ)-invexity requirements will be placed on certain combinations of these functions.

THEOREM 4.4 (Weak Duality). *Let x and $z \equiv (y, u, v, w, \alpha, \beta, \gamma)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is $(\eta, \bar{\rho})$-pseudoinvex at y;*
(ii) *for each $j \in J_+ \equiv J_+(v)$, $\mathcal{C}_j(\cdot, \gamma)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at y;*
(iii) *for each $k \in K_* \equiv K_*(w)$, $\mathcal{D}_k(\cdot, w)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at y;*
(iv) *$\bar{\rho} + \sum_{j \in J_+} v_j \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$;*

(b) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is $(\eta, \bar{\rho})$-pseudoinvex at y;*
(ii) *$\mathcal{C}(\cdot, v, \gamma)$ is $(\eta, \hat{\rho})$-quasiinvex at y;*
(iii) *for each $k \in K_*$, $D_k(\cdot, w)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at y;*
(iv) *$\bar{\rho} + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$;*

(c) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is $(\eta, \bar{\rho})$-pseudoinvex at y;*
(ii) *for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at y;*
(iii) *$\mathcal{D}(\cdot, w)$ is $(\eta, \breve{\rho})$-quasiinvex at y;*
(iv) *$\bar{\rho} + \sum_{j \in J_+} v_j \hat{\rho}_j + \breve{\rho} \geqq 0$;*

(d) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is $(\eta, \bar{\rho})$-pseudoinvex at y;*
(ii) *$\mathcal{C}(\cdot, v, \gamma)$ is $(\eta, \hat{\rho})$-quasiinvex at y;*
(iii) *$\mathcal{D}(\cdot, w)$ is $(\eta, \breve{\rho})$-quasiinvex at y;*
(iv) *$\bar{\rho} + \hat{\rho} + \breve{\rho} \geqq 0$;*

(e) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is $(\eta, \bar{\rho})$-pseudoinvex at y;*
(ii) *$\mathcal{F}(\cdot, v, w, \gamma)$ is $(\eta, \hat{\rho})$-quasiinvex at y;*
(iii) *$\bar{\rho} + \hat{\rho} \geqq 0$.*

Then $\varphi(x) \nleq \theta(z)$.

Proof. (a) Combining (4.3) with (4.9) and (4.10), which are valid for the present case due to our assumptions in (ii) and (iii), and using (iv), we obtain

$$\left\langle \sum_{i=1}^{p} u_i \{D_i^{\circ}(y, \beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^{\circ}(y, \alpha)[\nabla g_i(y) - B_i^T \beta^i]\}, \eta(x, y) \right\rangle$$
$$\geqq \left(\sum_{j \in J_+} v_j \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \right) \|x - y\|^2 \geqq -\bar{\rho} \|x - y\|^2,$$

which in view of (i) implies that $\mathcal{E}(x, y, u, \alpha, \beta) \geqq \mathcal{E}(y, y, u, \alpha, \beta) = 0$, where the equality follows from the definitions of $N_i^{\circ}(y, \alpha)$ and $D_i^{\circ}(y, \beta)$, $i \in \underline{p}$.

Now using this inequality and keeping in mind that $u > 0$, $N_i^\circ(y,\alpha) \geqq 0$ and $D_i^\circ(y,\beta) > 0$, $i \in \underline{p}$, we have

$$\begin{aligned}
0 &\leqq \sum_{i=1}^{p} u_i \{D_i^\circ(y,\beta)[f_i(x) + \langle \alpha^i, A_i x\rangle] - N_i^\circ(y,\alpha)[g_i(x) - \langle \beta^i, B_i x\rangle]\} \\
&\leqq \sum_{i=1}^{p} u_i \{D_i^\circ(y,\beta)[f_i(x) + \|\alpha^i\|_{a(i)}^* \\
&\quad \times \|A_i x\|_{a(i)}] - N_i^\circ(y,\alpha)[g_i(x) - \|\beta^i\|_{b(i)}^* \|B_i x\|_{b(i)}]\} \text{ (by Lemma 3.1)} \\
&\leqq \sum_{i=1}^{p} u_i \{D_i^\circ(y,\beta)[f_i(x) + \|A_i x\|_{a(i)}] - N_i^\circ(y,\alpha)[g_i(x) - \|B_i x\|_{b(i)}]\} \\
&\quad \text{(by (4.6))}.
\end{aligned}$$

As shown in the proof of Theorem 3.1, this inequality leads to the desired conclusion that $\varphi(x) \nleq \theta(z)$.

(b)–(e) The proofs are similar to that of part (a). □

THEOREM 4.5 (Weak Duality). *Let x and $z \equiv (y, u, v, w, \alpha, \beta, \gamma)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following twelve sets of hypotheses is satisfied:*

(a) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is prestrictly $(\eta, \bar\rho)$-quasiinvex at y;*
(ii) *for each $j \in J_+ \equiv J_+(v)$, $\mathcal{C}_j(\cdot, \gamma)$ is $(\eta, \hat\rho_j)$-quasiinvex at y;*
(iii) *for each $k \in K_* \equiv K_*(w)$, $\mathcal{D}_k(\cdot, w)$ is $(\eta, \breve\rho_k)$-quasiinvex at y;*
(iv) *$\bar\rho + \sum_{j\in J_+} v_j \hat\rho_j + \sum_{k\in K_*} \breve\rho_k > 0$;*

(b) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is prestrictly $(\eta, \bar\rho)$-quasiinvex at y;*
(ii) *$\mathcal{C}(\cdot, v, \gamma)$ is $(\eta, \hat\rho)$-quasiinvex at y;*
(iii) *for each $k \in K_*$, $\mathcal{D}_k(., w)$ is $(\eta, \breve\rho_k)$-quasiinvex at y;*
(iv) *$\bar\rho + \hat\rho + \sum_{k\in K_*} \breve\rho_k > 0$;*

(c) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is prestrictly $(\eta, \bar\rho)$-quasiinvex at y;*
(ii) *for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ is $(\eta, \hat\rho_j)$-quasiinvex at y;*
(iii) *$\mathcal{D}(\cdot, w)$ is $(\eta, \breve\rho)$-quasiinvex at y;*
(iv) *$\bar\rho + \sum_{j\in J_+} v_j \hat\rho_j + \breve\rho > 0$;*

(d) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is prestrictly $(\eta, \bar\rho)$-quasiinvex at y;*
(ii) *$\mathcal{C}(\cdot, v, \gamma)$ is $(\eta, \hat\rho)$-quasiinvex at y;*
(iii) *$\mathcal{D}(\cdot, w)$ is $(\eta, \breve\rho)$-quasiinvex at y;*
(iv) *$\bar\rho + \hat\rho + \breve\rho > 0$;*

(e) (i) *$\mathcal{E}(\cdot, y, u, \alpha, \beta)$ is prestrictly $(\eta, \bar\rho)$-quasiinvex at y;*
(ii) *$\mathcal{F}(\cdot, v, w, \gamma)$ is $(\eta, \tilde\rho)$-quasiinvex at y;*
(iii) *$\bar\rho + \tilde\rho > 0$;*

(f) (i) $\mathcal{E}(\cdot, y, u, \alpha, \beta)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* y*;*
(ii) *for each* $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ *is strictly* $(\eta, \hat{\rho}_j)$*-pseudoinvex at* y*;*
(iii) *for each* $k \in K_*$, $\mathcal{D}_k(\cdot, w)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* y*;*
(iv) $\bar{\rho} + \sum_{j\in J_+} v_j \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k \geqq 0$*;*

(g) (i) $\mathcal{E}(\cdot, y, u, \alpha, \beta)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* y*;*
(ii) $\mathcal{C}(\cdot, v, \gamma)$ *is strictly* $(\eta, \hat{\rho})$*-pseudoinvex at* y*;*
(iii) *for each* $k \in K_*$, $\mathcal{D}_k(\cdot, w)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* y*;*
(iv) $\bar{\rho} + \hat{\rho} + \sum_{k\in K_*} \breve{\rho}_k \geqq 0$*;*

(h) (i) $\mathcal{E}(\cdot, y, u, \alpha, \beta)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* y*;*
(ii) *for each* $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* y*;*
(iii) *for each* $k \in K_*$, $\mathcal{D}_k(\cdot, w)$ *is strictly* $(\eta, \breve{\rho}_k)$*-pseudoinvex at* y*;*
(iv) $\bar{\rho} + \sum_{j\in J_+} v_j \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k \geqq 0$*;*

(i) (i) $\mathcal{E}(\cdot, y, u, \alpha, \beta)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* y*;*
(ii) *for each* $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* y*;*
(iii) $\mathcal{D}(\cdot, w)$ *is strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* y*;*
(iv) $\bar{\rho} + \sum_{j\in J_+} v_j \hat{\rho}_j + \breve{\rho} \geqq 0$*;*

(j) (i) $\mathcal{E}(\cdot, y, u, \alpha, \beta)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* y*;*
(ii) $\mathcal{C}(\cdot, v, \gamma)$ *is strictly* $(\eta, \hat{\rho})$*-pseudoinvex at* y*;*
(iii) $\mathcal{D}(\cdot, w)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* y*;*
(iv) $\bar{\rho} + \hat{\rho} + \breve{\rho} \geqq 0$*;*

(k) (i) $\mathcal{E}(\cdot, y, u, \alpha, \beta)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* y*;*
(ii) $\mathcal{C}(\cdot, v, \gamma)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* y*;*
(iii) $\mathcal{D}(\cdot, w)$ *strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* y*;*
(iv) $\bar{\rho} + \hat{\rho} + \breve{\rho} \geqq 0$*;*

(l) (i) $\mathcal{E}(\cdot, y, u, \alpha, \beta)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* y*;*
(ii) $\mathcal{F}(\cdot, v, w, \gamma)$ *is strictly* $(\eta, \tilde{\rho})$*-pseudoinvex at* y*;*
(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then $\varphi(x) \nleq \theta(z)$.

Proof. (a) Because of our assumptions specified in (ii) and (iii), (4.9) and (4.10) remain valid for the present case. From (4.3), (4.9), (4.10), and (iv) we deduce that

$$\left\langle \sum_{i=1}^{p} u_i \{D_i^\circ(y, \beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^\circ(y, \alpha)[\nabla g_i(y) - B_i^T \beta^i]\}, \eta(x, y) \right\rangle$$
$$\geqq \left(\sum_{j\in J_+} v_j \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k \right) \|x - y\|^2 > -\bar{\rho}\|x - y\|^2,$$

which in view of (i) implies that $\mathcal{E}(x, y, u, \alpha, \beta) \geqq \mathcal{E}(y, y, u, \alpha, \beta) = 0$, where the equality follows from the definitions of $N_i^\circ(y, \alpha)$ and $D_i^\circ(y, \beta)$, $i \in \underline{p}$. As shown in the proof of Theorem 4.4, this inequality leads to the conclusion that $\varphi(x) \nleq \theta(z)$.

(b)–(e) The proofs are similar to that of part (a).

(f) As shown in the proof of part (a) of Theorem 4.1, for each $j \in J_+$, we have $G_j(x) + \langle \gamma^j, C_j x \rangle \leqq G_j(y) + \langle \gamma^j, C_j y \rangle$, which by (ii) implies that

$$\langle \nabla G_j(y) + C_j^T \gamma^j, \eta(x, y) \rangle < -\hat{\rho}_j \|x - y\|^2.$$

As $v_j \geqq 0$ for each $j \in \underline{q}$, and $v_j = 0$ for each $j \in \underline{q} \backslash J_+$, the above inequalities yield

$$\left\langle \sum_{j=1}^{q} v_j [\nabla G_j(y) + C_j^T \gamma^j], \eta(x, y) \right\rangle < -\sum_{j \in J_+} v_j \hat{\rho}_j \|x - y\|^2.$$

Now combining this inequality with (4.10) (which is valid for the present case because of (iii)) and (4.3), and using the primal feasibility of x and (iv), we obtain

$$\left\langle \sum_{i=1}^{p} u_i \{ D_i^\circ(y, \beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^\circ(y, \alpha)[\nabla g_i(y) - B_i^T \beta^i] \}, \eta(x, y) \right\rangle$$
$$> \Big(\sum_{j \in J_+} v_j \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \Big) \|x - y\|^2 \geqq -\bar{\rho} \|x - y\|^2,$$

which in view of (i) implies that $\mathcal{E}(x, y, u, \alpha, \beta) \geqq \mathcal{E}(y, y, u, \alpha, \beta) = 0$. As seen in the proof of Theorem 4.4, this leads to the conclusion that $\varphi(x) \nleq \theta(z)$.

(g)–(l) The proofs are similar to that of part (f). □

THEOREM 4.6 (Weak Duality). *Let x and $z \equiv (y, u, v, w, \alpha, \beta, \gamma)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at y;*
(ii) *for each $j \in J_+ \equiv J_+(v)$, $\mathcal{C}_j(\cdot, \gamma)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at y;*
(iii) *for each $k \in K_* \equiv K_+(w)$, $\mathcal{D}_k(., w)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at y;*
(iv) $\rho^\circ + \sum_{j \in J_+} v_j \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$, *where* $\rho^\circ = \sum_{i=1}^{p} u_i \bar{\rho}_i$*;*

(b) (i) *for each $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*;*
(ii) *$\mathcal{C}(\cdot, v, \gamma)$ is $(\eta, \hat{\rho})$-quasiinvex at y;*
(iii) *for each $k \in K_*$, $\mathcal{D}_k(\cdot, w)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at y;*
(iv) $\rho^\circ + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$*;*

(c) (i) *for each $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at y;*
(ii) *for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at y;*

(iii) $\mathcal{D}(\cdot, w)$ *is* $(\eta, \breve{\rho})$-*quasiinvex at* y;
(iv) $\rho^\circ + \sum_{j\in J_+} v_j \hat{\rho}_j + \breve{\rho} \geqq 0$;

(d) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is strictly* $(\eta, \bar{\rho}_i)$-*pseudoinvex at* y;
(ii) $\mathcal{C}(\cdot, v, \gamma)$ *is* $(\eta, \hat{\rho})$-*quasiinvex at* y;
(iii) $\mathcal{D}(\cdot, w)$ *is* $(\eta, \breve{\rho})$-*quasiinvex at* y;
(iv) $\rho^\circ + \hat{\rho} + \breve{\rho} \geqq 0$;

(e) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is strictly* $(\eta, \bar{\rho}_i)$-*pseudoinvex at* y;
(ii) $\mathcal{F}(\cdot, v, w, \gamma)$ *is* $(\eta, \tilde{\rho})$-*quasiinvex at* y;
(iii) $\rho^\circ + \tilde{\rho} \geqq 0$.

Then $\varphi(x) \nleq \theta(z)$.

Proof. (a) Suppose to the contrary that $\varphi(x) \leqslant \theta(z)$, that is,

$$\frac{f_i(x) + \|A_i x\|_{a(i)}}{g_i(x) - \|B_i x\|_{b(i)}} \leqq \frac{f_i(y) + \langle \alpha^i, A_i y\rangle}{g_i(y) - \langle \beta^i, B_i y\rangle} \quad \text{for each } i \in \underline{p}.$$

and

$$\frac{f_m(x) + \|A_m x\|_{a(m)}}{g_m(x) - \|B_m x\|_{b(m)}} < \frac{f_m(y) + \langle \alpha^m, A_m y\rangle}{g_m(y) - \langle \beta^m, B_m y\rangle} \quad \text{for some } m \in \underline{p}.$$

Hence, for each $i \in \underline{p}$, we have

$$D_i^\circ(y, \beta)[f_i(x) + \|A_i x\|_{a(i)}] - N_i^\circ(y, \alpha)[g_i(x) - \|B_i x\|_{b(i)}] \leqq 0. \tag{4.12}$$

Since for each $i \in \underline{p}$,

$$\begin{aligned}
\mathcal{E}_i(x, y, \alpha, \beta) &= D_i^\circ(y, \beta)[f_i(x) + \langle \alpha^i, A_i x\rangle] - N_i^\circ(y, \beta)[g_i(x) - \langle \beta^i, B_i x\rangle] \\
&\leqq D_i^\circ(y, \beta)[f_i(x) + \|\alpha^i\|^*_{a(i)} \|A_i x\|_{a(i)}] \\
&\quad - N_i^\circ(y, \alpha)[g_i(x) - \|\beta^i\|^*_{b(i)} \|B_i x\|_{b(i)}] \\
&\quad \text{(by Lemma 3.1)} \\
&\leqq D_i^\circ(y, \beta)[f_i(x) + \|A_i x\|_{a(i)}] - N_i^\circ(y, \alpha)[g_i(x) - \|B_i x\|_{b(i)}] \\
&\quad \text{(by (4.6))} \\
&\leqq 0 \text{ (by (4.12))} \\
&= D_i^\circ(y, \beta)[f_i(y) + \langle \alpha^i, A_i y\rangle] - N_i^\circ(y, \alpha)[g_i(y) - \langle \beta^i, B_i y\rangle] \\
&\quad \text{(by the definitions of } D_i^\circ(y, \beta) \text{ and } N^\circ(y, \alpha),\ i \in \underline{p}) \\
&= \mathcal{E}_i(y, y, \alpha, \beta),
\end{aligned}$$

it follows from (i) that

$$\begin{aligned}
&\langle D_i^\circ(y, \beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^\circ(y, \alpha)[\nabla g_i(y) - B_i^T \beta^i], \eta(x, y)\rangle \\
&\quad < -\bar{\rho}_i \|x - y\|^2.
\end{aligned}$$

Because $u > 0$, we deduce from the above inequalities that

$$\left\langle \sum_{i=1}^{p} u_i \{D_i^{\circ}(y,\beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i(y,\alpha)[\nabla g_i(y) - B_i^T \beta^i]\}, \right.$$
$$\left. \eta(x,y) \right\rangle < -\sum_{i=1}^{p} u_i \bar{\rho}_i \|x-y\|^2. \tag{4.13}$$

As shown in the proof of Theorem 4.1, our assumptions in (ii) and (iii) lead to (4.9) and (4.10), respectively, which when combined with (4.3) and (iv) yield

$$\left\langle \sum_{i=1}^{p} u_i \{D_i(y,\beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^{\circ}(y,\alpha)[\nabla g_i(y) - B_i^T \beta^i]\}, \eta(x,y) \right\rangle$$
$$\geqq \Big(\sum_{j\in J_+} v_j \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k \Big) \|x-y\|^2 \geqq -\sum_{i=1}^{p} u_i \bar{\rho}_i \|x-y\|^2,$$

which contradicts (4.13). Therefore, we conclude that $\varphi(x) \nleq \theta(z)$.

(b)–(e) The proofs are similar to that of part (a). □

THEOREM 4.7 (Weak Duality). *Let x and $(y,u,v,w,\alpha,\beta,\gamma)$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following twelve sets of hypotheses is satisfied:*

(a) (i) *for each $i \in \underline{p}$, $\mathcal{E}_i(\cdot,y,\alpha,\beta)$ is $(\eta,\bar{\rho}_i)$-quasiinvex at y;*
(ii) *for each $j \in \underline{J_+} \equiv J_+(v)$, $\mathcal{C}_j(\cdot,\gamma)$ is $(\eta,\hat{\rho}_j)$-quasiinvex at y;*
(iii) *for each $k \in K_* \equiv K_*(w)$, $\mathcal{D}_k(\cdot,w)$ is $(\eta,\breve{\rho}_k)$-quasiinvex at y;*
(iv) $\rho^{\circ} + \sum_{j\in J_+} v_j \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k > 0$, *where* $\rho^{\circ} = \sum_{i=1}^{p} u_i \bar{\rho}_i$;

(b) (i) *for each $i \in \underline{p}$, $\mathcal{E}_i(\cdot,y,\alpha,\beta)$ is $(\eta,\bar{\rho}_i)$-quasiinvex at y;*
(ii) *$\mathcal{C}(\cdot,v,\gamma)$ is $(\eta,\hat{\rho})$-quasiinvex at y;*
(iii) *for each $k \in K_*$, $\mathcal{D}_k(\cdot,w)$ is $(\eta,\breve{\rho}_k)$-quasiinvex at y;*
(iv) $\rho^{\circ} + \hat{\rho} + \sum_{k\in K_*} \breve{\rho}_k > 0$;

(c) (i) *for each $i \in \underline{p}$, $\mathcal{E}_i(\cdot,y,\alpha,\beta)$ is $(\eta,\bar{\rho}_i)$-quasiinvex at y;*
(ii) *for each $j \in \underline{J_+}$, $\mathcal{C}_j(\cdot,\gamma)$ is $(\eta,\hat{\rho}_j)$-quasiinvex at y;*
(iii) *$\mathcal{D}(\cdot,w)$ is $(\eta,\breve{\rho})$-quasiinvex at y;*
(iv) $\rho^{\circ} + \sum_{j\in J_+} v_j \hat{\rho}_j + \breve{\rho} > 0$;

(d) (i) *for each $i \in \underline{p}$, $\mathcal{E}_i(\cdot,y,\alpha,\beta)$ is $(\eta,\bar{\rho}_i)$-quasiinvex at y;*
(ii) *$\mathcal{C}(\cdot,v,\gamma)$ is $(\eta,\hat{\rho})$-quasiinvex at y;*
(iii) *$\mathcal{D}(\cdot,w)$ is $(\eta,\breve{\rho})$-quasiinvex at y;*
(iv) $\rho^{\circ} + \hat{\rho} + \breve{\rho} > 0$;

(e) (i) *for each $i \in \underline{p}$, $\mathcal{E}_i(\cdot,y,\alpha,\beta)$ is $(\eta,\bar{\rho}_i)$-quasiinvex at y;*
(ii) *$\mathcal{F}(\cdot,v,w,\gamma)$ is $(\eta,\tilde{\rho})$-quasiinvex at y;*
(iii) $\rho^{\circ} + \tilde{\rho} > 0$;

(f) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* y*;*
(ii) *for each* $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ *is strictly* $(\eta, \hat{\rho}_j)$*-pseudoinvex at* y*;*
(iii) *for each* $k \in K_*$, $\mathcal{D}_k(\cdot, w)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* y*;*
(iv) $\rho^\circ + \sum_{j\in J_+} v_j \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k \geqq 0$*;*

(g) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* y*;*
(ii) $\mathcal{C}(\cdot, v, \gamma)$ *is strictly* $(\eta, \hat{\rho})$*-pseudoinvex at* y*;*
(iii) *for each* $k \in K_*$, $\mathcal{D}_k(\cdot, w)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* y*;*
(iv) $\rho^\circ + \hat{\rho} + \sum_{k\in K_*} \breve{\rho}_k \geqq 0$*;*

(h) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* y*;*
(ii) *for each* $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* y*;*
(iii) *for each* $k \in K_*$, $\mathcal{D}_k(\cdot, w)$ *is strictly* $(\eta, \breve{\rho}_k)$*-pseudoinvex at* y*;*
(iv) $\rho^\circ + \sum_{j\in J_+} v_j \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k \geqq 0$*;*

(i) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* y*;*
(ii) *for each* $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* y*;*
(iii) $\mathcal{D}(\cdot, w)$ *is strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* y*;*
(iv) $\rho^\circ + \sum_{j\in J_+} v_j \hat{\rho}_j + \breve{\rho} \geqq 0$*;*

(j) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* y*;*
(ii) $\mathcal{C}(\cdot, v, \gamma)$ *is strictly* $(\eta, \hat{\rho})$*-pseudoinvex at* y*;*
(iii) $\mathcal{D}(\cdot, w)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* y*;*
(iv) $\rho^\circ + \hat{\rho} + \breve{\rho} \geqq 0$*;*

(k) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* y*;*
(ii) $\mathcal{C}(\cdot, v, \gamma)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* y*;*
(iii) $\mathcal{D}(\cdot, w)$ *strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* y*;*
(iv) $\rho^\circ + \hat{\rho} + \breve{\rho} \geqq 0$*;*

(l) (i) *for each* $i \in \underline{p}$, $\mathcal{E}_i(\cdot, y, \alpha, \beta)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* y*;*
(ii) $\mathcal{F}(\cdot, v, w, \gamma)$ *is strictly* $(\eta, \tilde{\rho})$*-pseudoinvex at* y*;*
(iii) $\rho^\circ + \tilde{\rho} \geqq 0$.

Then $\varphi(x) \nleq \theta(z)$.

Proof. (a) Suppose to the contrary that $\varphi(x) \leqslant \theta(z)$. As shown in the proof of Theorem 4.6, this supposition leads to the inequalities $\mathcal{E}_i(x, y, \alpha, \beta) \leqq \mathcal{E}_i(y, y, \alpha, \beta)$ for each $i \in \underline{p}$, and $\mathcal{E}_m(x, y, \alpha, \beta) < \mathcal{E}_m(y, y, \alpha, \beta)$ for some $m \in \underline{p}$. In view of (i), this implies that for each $i \in \underline{p}$,

$$\langle D_i^\circ(y, \beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^\circ(y, \alpha)[\nabla g_i(y) - B_i^T \beta^i], \eta(x, y)\rangle \leqq -\bar{\rho}_i \|x - y\|^2.$$

Since $u > 0$, the above inequalities yield

$$\left\langle \sum_{i=1}^{p} u_i \{D_i^\circ(y, \beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^\circ(y, \alpha)[\nabla g_i(y) - B_i^T \beta^i]\}, \eta(x, y) \right\rangle$$
$$\leqq -\sum_{i=1}^{p} u_i \bar{\rho}_i \|x - y\|^2. \tag{4.14}$$

As shown earlier, our assumptions in (ii) and (iii) lead to (4.9) and (4.10), respectively, which when combined with (4.3) and (iv) yield

$$\left\langle \sum_{i=1}^{p} u_i\{D_i^{\circ}(y,\beta)[\nabla f_i(y)+A_i^T\alpha^i]-N_i^{\circ}(y,\alpha)[\nabla g_i(y)-B_i^T\beta^i]\},\eta(x,y)\right\rangle$$
$$>-\sum_{i=1}^{p} u_i\bar{\rho}_i\|x-y\|^2.$$

which contradicts (4.14). Hence $\varphi(x)\nleq\theta(z)$.

(b)–(l) : The proofs are similar to that of part (a). □

THEOREM 4.8 (Strong Duality). *Let x^* be a normal efficient solution of (P) and assume that any one of the thirty four sets of conditions set forth in Theorems 4.4–4.7 is satisfied for all feasible solutions of (DII). Then there exist $u^*\in U$, $v^*\in\mathbb{R}^q_+$, $w^*\in\mathbb{R}^r$, $\alpha^{*i}\in\mathbb{R}^{\ell_i}$, $\beta^{*i}\in\mathbb{R}^{m_i}$, $i\in\underline{p}$, and $\gamma^{*j}\in\mathbb{R}^{n_j}$, $j\in\underline{q}$, such that $z^*\equiv(x^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ is an efficient solution of (DII) and $\varphi(x^*)=\theta(z^*)$.*

Proof. The proof is similar to that of Theorem 3.3. □

THEOREM 4.9 (Strict Converse Duality). *Let x^* be an efficient solution of (P) and let $\tilde{z}\equiv(\tilde{x},\tilde{u},\tilde{v},\tilde{w},\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ be a feasible solution of (DII) such that*

$$\sum_{i=1}^{p}\tilde{u}_i\{D_i^{\circ}(\tilde{x},\tilde{\beta})[f_i(x^*)+\|A_ix^*\|_{a(i)}]-N_i^{\circ}(\tilde{x},\tilde{\alpha})[g_i(x^*)-\|B_ix^*\|_{b(i)}]\}\leqq 0. \tag{4.15}$$

Furthermore, assume that any one of the five sets of hypotheses specified in Theorem 4.4 is satisfied for all feasible solutions of (DII), and that $\mathcal{E}(\cdot,\tilde{x},\tilde{u},\tilde{x},\tilde{\alpha},\tilde{\beta})$ is strictly $(\eta,\bar{\rho})$-pseudoinvex at $\tilde{x}$. Then $\tilde{x}=x^*$ and $\varphi(x^*)=\theta(\tilde{x})$.

Proof. (a) Suppose, contrary to what we want to show that $\tilde{x}\neq x^*$. Now proceeding as in the proof of part (a) of Theorem 4.4 (with x replaced by x^* and z by $\tilde{z}$), we obtain

$$\left\langle \sum_{i=1}^{p}\tilde{u}_i\{D_i^{\circ}(\tilde{x},\tilde{\beta})[\nabla f_i(\tilde{x})+A_i^T\tilde{\alpha}^i]-N_i^{\circ}(\tilde{x},\tilde{\alpha})[\nabla g_i(\tilde{x})-B_i^T\tilde{\beta}^i]\},\eta(x^*,\tilde{x})\right\rangle$$
$$\geqq\left(\sum_{j\in J_+}\tilde{v}_j\hat{\rho}_j+\sum_{k\in K_*}\breve{\rho}_k\right)\|x^*-\tilde{x}\|^2\geqq-\bar{\rho}\|x^*-\tilde{x}\|^2.$$

In view of our strict $(\eta,\bar{\rho})$-pseudoinvexity assumption, this inequality implies that $\mathcal{E}(x^*,\tilde{x},\tilde{u},\tilde{\alpha},\tilde{\beta})>\mathcal{E}(\tilde{x},\tilde{x},\tilde{u},\tilde{\alpha},\tilde{\beta})=0$, where the equality follows from the definitions of $D_i^\circ(\tilde{x},\tilde{\beta})$ and $N_i^\circ(\tilde{x},\tilde{\alpha})$, $i\in\underline{p}$. This obviously contradicts (4.15), and hence we conclude that $\tilde{x}=x^*$.

(b)–(e) The proofs are similar to that of part (a). □

5. Duality Model III

In this section, we discuss several families of duality results under various generalized (η,ρ)-invexity hypotheses imposed on certain combinations of the problem functions. This is accomplished by employing a certain type of partitioning scheme which was originally proposed in [13] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\{J_0,J_1,\dots,J_m\}$ and $\{K_0,K_1,\dots,K_m\}$ be partitions of the index sets $\underline{q}$ and $\underline{r}$, respectively; thus, $J_\mu\subset\underline{q}$ for each $\mu\in\underline{m}\cup\{0\}$, $J_\mu\cap J_\nu=\emptyset$ for each $\mu,\nu\in\underline{m}\cup\{0\}$ with $\mu\neq\nu$, and $\bigcup_{\mu=0}^m J_\mu=\underline{q}$. Obviously, similar properties hold for $\{K_0,K_1,\dots,K_m\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{q}$ and $\underline{r}$, respectively, then $m=\max\{m_1,m_2\}$ and $J_\mu=\emptyset$ or $K_\mu=\emptyset$ for $\mu>\min\{m_1,m_2\}$.

In addition, we use the real-valued functions $\Phi_i(\cdot,\bar{x},v,w,\alpha,\beta,\gamma)$, $i\in\underline{p}$, $\Phi(\cdot,\bar{x},u,v,w,\alpha,\beta,\gamma)$, $\Lambda_t(\cdot,v,w)$, and $\Lambda_t^\circ(\cdot,v,w,\gamma)$ defined, for fixed $\bar{x},u,v,w,\alpha,\beta$, and γ, on X as follows:

$$\begin{aligned}\Phi_i(x,y,v,w,\alpha,\beta,\gamma)=D_i^\circ(y,\beta)\Big[&f_i(x)+\langle\alpha^i,A_ix\rangle\\&+\sum_{j\in J_0}v_j[G_j(x)+\langle\gamma^j,C_jx\rangle+\sum_{k\in K_0}w_kH_k(x)\Big]\\&-[N_i^\circ(y,\alpha)+\Lambda_0^\circ(\bar{x},v,w,\gamma)]\\&\times[g_i(x)-\langle\beta^i,B_ix\rangle],\quad i\in\underline{p},\end{aligned}$$

$$\begin{aligned}\Phi(x,y,u,v,w,\alpha,\beta,\gamma)=\sum_{i=1}^p u_i\Big\{D_i^\circ(y,\beta)\Big[&f_i(x)+\langle\alpha^i,A_ix\rangle\\&+\sum_{j\in J_0}v_j[G_j(x)+\langle\gamma^j,C_jx\rangle+\sum_{k\in K_0}w_kH_k(x)\Big]\\&-[N_i^\circ(y,\alpha)+\Lambda_0^\circ(y,v,w,\gamma)]\\&\times[g_i(x)-\langle\beta^i,B_ix\rangle]\Big\},\end{aligned}$$

$$\Lambda_t(x,v,w)=\sum_{j\in J_t}v_j[G_j(x)+\|C_jx\|_{c(j)}]+\sum_{k\in K_t}w_kH_k(x),\quad t\in\underline{m}\cup\{0\},$$

$$\Lambda_t^\circ(x,v,w,\gamma)=\sum_{j\in J_t}v_j[G_j(x)+\langle\gamma^j,C_jx\rangle]+\sum_{k\in K_t}w_kH_k(x),\quad t\in\underline{m}\cup\{0\}.$$

Making use of the sets and functions defined above, we can state our general duality models as follows:

(CIII)

$$\text{Maximize}\left(\frac{f_1(y)+\|A_1 y\|_{a(1)}+\Lambda_0(y,v,w)}{g_1(y)-\|B_1 y\|_{b(1)}},\ \ldots,\ \frac{f_p(y)+\|A_p y\|_{a(p)}+\Lambda_0(y,v,w)}{g_p(y)-\|B_p y\|_{b(p)}}\right)$$

subject to

$$\sum_{i=1}^{p} u_i\Big\{D_i(y)\Big\{\nabla f_i(y)+A_i^T\alpha^i+\sum_{j\in J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_0} w_k\nabla H_k(y)\Big\} -[N_i(y)+\Lambda_0(y,v,w)][\nabla g_i(y)-B_i^T\beta^i]\Big\} + \sum_{j\in \underline{q}\setminus J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in \underline{r}\setminus K_0} w_k\nabla H_k(y)=0, \tag{5.1}$$

$$\sum_{j\in J_t} v_j[G_j(y)+\|C_j y\|_{c(j)}]+\sum_{k\in K_t} w_k H_k(y)\geqq 0,\ \ t\in \underline{m}\cup\{0\}, \tag{5.2}$$

$$\|\alpha^i\|^*_{a(i)}\leqq 1,\quad \|\beta^i\|^*_{b(i)}\leqq 1,\quad i\in\underline{p}, \tag{5.3}$$

$$\|\gamma^j\|^*_{c(j)}\leqq 1,\ \ j\in\underline{q}, \tag{5.4}$$

$$\langle\alpha^i, A_i y\rangle=\|A_i y\|_{a(i)},\ \ \langle\beta^i, B_i y\rangle=\|B_i y\|_{b(i)},\ \ i\in\underline{p}, \tag{5.5}$$

$$\langle\gamma^j, C_j y\rangle=\|C_j y\|_{c(j)},\quad j\in\underline{q}. \tag{5.6}$$

$$y\in X,\ u\in U,\ v\in\mathbb{R}^q_+,\ w\in\mathbb{R}^r,\ \alpha^i\in\mathbb{R}^{\ell_i},\ \beta^i\in\mathbb{R}^{m_i},\ i\in\underline{p},\ \gamma^j\in\mathbb{R}^{n_j},\ j\in\underline{q}; \tag{5.7}$$

($\tilde{\text{C}}$III)

$$\text{Maximize}\left(\frac{f_1(y)+\|A_1 y\|_{a(1)}+\Lambda_0(y,v,w)}{g_1(y)-\|B_1 y\|_{b(1)}},\ \ldots,\ \frac{f_p(y)+\|A_p y\|_{a(p)}+\Lambda_0(y,v,w)}{g_p(y)-\|B_p y\|_{b(p)}}\right)$$

subject to (5.2)–(5.7) and

$$\Big\langle \sum_{i=1}^{p} u_i \Big\{ D_i(y) \Big\{ \nabla f_i(y) + A_i^T \alpha^i + \sum_{j\in J_0} v_j [\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k\in K_0} w_k \nabla H_k(y) \Big\} - [N_i(y) + \Lambda_0(y, v, w)][\nabla g_i(y) - B_i^T \beta^i] \Big\} + \sum_{j\in \underline{q}\setminus J_0} v_j [\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k\in \underline{r}\setminus K_0} w_k \nabla H_k(y), \eta(x, y) \Big\rangle \geqq 0 \quad \text{for all } x \in \mathbb{F}, \tag{5.8}$$

where η is a function from $X \times X$ to $\mathbb{R}^n$;

(DIII)

$$\text{Maximize } \left(\frac{f_1(y) + \langle \alpha^1, A_1 y\rangle + \Lambda_0^{\circ}(y, v, w, \gamma)}{g_1(y) - \langle \beta^1, B_1 y\rangle}, \ldots, \frac{f_p(y) + \langle \alpha^p, A_p y\rangle + \Lambda_0^{\circ}(y, v, w, \gamma)}{g_p(y) - \langle \beta^p, B_p y\rangle} \right)$$

subject to (5.7) and

$$\sum_{i=1}^{p} u_i \Big\{ D_i^{\circ}(y, \beta) \Big\{ \nabla f_i(y) + A_i^T \alpha^i + \sum_{j\in J_0} v_j [\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k\in K_0} w_k \nabla H_k(y) \Big\} - [N_i^{\circ}(y, \alpha) + \Lambda_0^{\circ}(y, v, w, \gamma)][\nabla g_i(y) - B_i^T \beta^i] \Big\} + \sum_{j\in \underline{q}\setminus J_0} v_j [\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k\in \underline{r}\setminus K_0} w_k \nabla H_k(y) = 0, \tag{5.9}$$

$$\sum_{j\in J_t} v_j [G_j(y) + \langle \gamma^j, C_j y\rangle] + \sum_{k\in K_t} w_k H_k(y) \geqq 0, \quad t \in \underline{m} \cup \{0\}, \tag{5.10}$$

$$\|\alpha^i\|_{a(i)}^* \leqq 1, \qquad \|\beta^i\|_{b(i)}^* \leqq 1, \quad i \in \underline{p}, \tag{5.11}$$

$$\|\gamma^j\|_{c(j)}^* \leqq 1, \quad j \in \underline{q}; \tag{5.12}$$

($\tilde{\text{D}}$III)

$$\text{Maximize}\quad \left(\frac{f_1(y)+\langle\alpha^1, A_1 y\rangle+\Lambda_0^\circ(y,v,w,\gamma)}{g_1(y)-\langle\beta^1, B_1 y\rangle}, \ldots, \frac{f_p(y)+\langle\alpha^p, A_p y\rangle+\Lambda_0^\circ(y,v,w,\gamma)}{g_p(y)-\langle\beta^p, B_p y\rangle}\right)$$

subject to (5.7), (5.10)–(5.12) and

$$\left\langle \sum_{i=1}^{p} u_i\Big\{D_i^\circ(y,\beta)\Big\{\nabla f_i(y)+A_i^T\alpha^i+\sum_{j\in J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j] + \sum_{k\in K_0} w_k\nabla H_k(y)\Big\} - [N_i^\circ(y,\alpha)+\Lambda_0^\circ(y,v,w,\gamma)][\nabla g_i(y)-B_i^T\beta^i]\Big\} + \sum_{j\in \underline{q}\setminus J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j] + \sum_{k\in\underline{r}\setminus K_0} w_k\nabla H_k(y), \eta(x,y)\right\rangle \geqq 0$$

for all $x\in\mathbb{F}$,

where η is a function from $X\times X$ to $\mathbb{R}^n$.

The remarks and observations made earlier about the relationships among (CI), ($\tilde{\text{C}}$I), (DI), and ($\tilde{\text{D}}$I) are, of course, also valid for (CIII), ($\tilde{\text{C}}$III), (DIII), and ($\tilde{\text{D}}$III). As in the preceding sections, we shall work with the reduced versions (DIII) and ($\tilde{\text{D}}$III), and, in particular, consider the pair (P)–(DIII).

The following two theorems show that (DIII) is a dual problem for (P).

THEOREM 5.1 (Weak Duality). *Let x and $z\equiv(y,u,v,w,\alpha,\beta,\gamma)$ be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

(a) (i) *$\Phi(\cdot,y,u,v,w,\alpha,\beta,\gamma)$ is prestrictly $(\eta,\bar\rho)$-quasiinvex at y;*
 (ii) *for each $t\in\underline{m}$, $\Lambda_t^\circ(\cdot,v,w,\gamma)$ is strictly $(\eta,\tilde\rho_t)$-pseudoinvex at y;*
 (iii) *$\bar\rho+\sum_{t=1}^m\tilde\rho_t\geqq 0$;*

(b) (i) *$\Phi(\cdot,y,u,v,w,\alpha,\beta,\gamma)$ is $(\eta,\bar\rho)$-pseudoinvex at y;*
 (ii) *for each $t\in\underline{m}$, $\Lambda_t^\circ(\cdot,v,w,\gamma)$ is $(\eta,\tilde\rho_t)$-quasiinvex at y;*
 (iii) *$\bar\rho+\sum_{t=1}^m\tilde\rho_t\geqq 0$;*

(c) (i) *$\Phi(\cdot,y,u,v,w,\alpha,\beta,\gamma)$ is prestrictly $(\eta,\bar\rho)$-quasiinvex at y;*
 (ii) *for each $t\in\underline{m}$, $\Lambda_t^\circ(\cdot,v,w,\gamma)$ is $(\eta,\tilde\rho_t)$-quasiinvex at y;*
 (iii) *$\bar\rho+\sum_{t=1}^m\tilde\rho_t> 0$;*

(d) (i) *$\Phi(\cdot,y,u,v,w,\alpha,\beta,\gamma)$ is prestrictly $(\eta,\bar\rho)$-quasiinvex at y;*
 (ii) *for each $t\in\underline{m_1}$, $\Lambda_t^\circ(\cdot,v,w,\gamma)$ is $(\eta,\tilde\rho_t)$-quasiinvex at y, and for each $t\in\underline{m_2}$, $\Lambda_t^\circ(\cdot,v,w,\gamma)$ is strictly $(\eta,\tilde\rho_t)$-pseudoinvex at y, where $\{\underline{m_1},\underline{m_2}\}$ is a partition of $\underline{m}$;*

(iii) $\bar{\rho}+\sum_{t=1}^{m}\tilde{\rho}_t \geqq 0$;
(iv) $\underline{m_2} \neq \emptyset$ *or* $\bar{\rho}+\sum_{t=1}^{m}\tilde{\rho}_t > 0$;

Then $\varphi(x) \nleq \xi(z)$, *where* $\xi=(\xi_1,\ldots,\xi_p)$ *is the objective function of* (*DIII*).

Proof. (a) It is clear that (5.9) can be expressed as follows:

$$\sum_{i=1}^{p} u_i\Big\{D_i^{\circ}(y,\beta)\Big\{\nabla f_i(y)+A_i^T\alpha^i+\sum_{j\in J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j] + \sum_{k\in K_0} w_k\nabla H_k(y)]\Big\} - [N_i^{\circ}(y,\alpha)+\Lambda_0^{\circ}(y,v,w,\gamma)][\nabla g_i(y)-B_i^T\beta^i]\Big\} + \sum_{t=1}^{m}\Big\{\sum_{j\in J_t} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_t} w_k\nabla H_k(y)\Big\}=0. \tag{5.13}$$

Since for each $t\in\underline{m}$,

$$\begin{aligned}
&\Lambda_t^{\circ}(x,v,w,\gamma)\\
&\quad=\sum_{j\in J_t} v_j[G_j(x)+\langle\gamma^j,C_jx\rangle]+\sum_{k\in K_t} w_kH_k(x)\\
&\quad\leqq\sum_{j\in J_t} v_j[G_j(x)+\|\gamma^j\|_{c(j)}^{*}\|C_jx\|_{c(j)}]\\
&\qquad+\sum_{k\in K_t} w_kH_k(x)\quad\text{(by Lemma 3.1 and nonnegativity of } v)\\
&\quad\leqq\sum_{j\in J_t} v_j[G_j(x)+\|C_jx\|_{c(j)}]\\
&\qquad+\sum_{k\in K_t} w_kH_k(x)\quad\text{(by (5.12) and nonnegativity of } v)\\
&\quad\leqq 0\quad\text{(by the primal feasibility of } x \text{ and nonnegativity of } v)\\
&\quad\leqq\sum_{j\in J_t} v_j[G_j(y)+\langle\gamma^j,C_jy\rangle]+\sum_{k\in K_t} w_kH_k(y)\quad\text{(by (5.10))}\\
&\qquad=\Lambda_t^{\circ}(y,v,w,\gamma),
\end{aligned}$$

it follows from (ii) that

$$\Big\langle\sum_{j\in J_t} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_t} w_k\nabla H_k(y),\,\eta(x,y)\Big\rangle < -\tilde{\rho}_t\|x-y\|^2.$$

Summing over t, we obtain

$$\Big\langle \sum_{t=1}^{m} \Big\{ \sum_{j\in J_t} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_t} w_k\nabla H_k(y)\Big\}, \eta(x,y)\Big\rangle$$
$$<-\sum_{t=1}^{m} \tilde{\rho}_t\|x-y\|^2. \tag{5.14}$$

Combining (5.13) and (5.14), and using (iii) we get

$$\Big\langle \sum_{i=1}^{p} u_i\Big\{D_i^\circ(y,\beta)\Big\{\nabla f_i(y)+A_i^T\alpha^i$$
$$+\sum_{j\in J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_0} w_k\nabla H_k(y)]\Big\}$$
$$-[N_i^\circ(y,\alpha)+\Lambda_0^\circ(y,v,w,\gamma)][\nabla g_i(y)-B_i^T\beta^i]\Big\}, \eta(x,y)\Big\rangle$$
$$>\sum_{t=1}^{m}\tilde{\rho}_t\|x-y\|^2 \geqq -\bar{\rho}\|x-y\|^2, \tag{5.15}$$

which by virtue of (i) implies that $\Phi(x,y,u,v,w,\alpha,\beta,\gamma) \geqq \Phi(y,y,u,v,w,\alpha,\beta,\gamma)=0$, where the equality follows from the definitions of $N_i^\circ(y,\alpha)$, $D_i^\circ(y,\beta)$, $i\in \underline{p}$, and $\Lambda_0^\circ(y,v,w,\gamma)$. Therefore, we have

$$\begin{aligned}
0 &\leqq \Phi(x,y,u,v,w,\alpha,\beta,\gamma)\\
&=\sum_{i=1}^{p} u_i\Big\{D_i^\circ(y,\beta)\Big\{f_i(x)+\langle\alpha^i, A_i x\rangle+\sum_{j\in J_0} v_j[G_j(x)+\langle\gamma^j, C_j x\rangle]\\
&\quad+\sum_{k\in K_0} w_k H_k(x)\Big\}-[N_i^\circ(y,\alpha)+\Lambda_0^\circ(y,v,w,\gamma)][g_i(x)-\langle\beta^i, B_i x\rangle]\Big\}\\
&\leqq \sum_{i=1}^{p} u_i\Big\{D_i^\circ(y,\beta)\Big\{f_i(x)+\|\alpha^i\|_{a(i)}^*\|A_i x\|_{a(i)}\\
&\quad+\sum_{j\in J_0} v_j[G_j(x)+\|\gamma^j\|_{c(j)}^*\|C_j x\|_{c(j)}]\Big\}-[N_i^\circ(y,\alpha)\\
&\quad+\Lambda_0^\circ(y,v,w,\gamma)][g_i(x)-\|\beta^i\|_{b(i)}^*\|B_i x\|_{b(i)}]\Big\}
\end{aligned}$$

(by Lemma 3.1, definition of $\Lambda_0^\circ(y,v,w,\gamma)$, (5.10), and primal feasibility of x)

$$\leqq \sum_{i=1}^{p} u_i \Big\{ D_i^\circ(y,\beta) \Big\{ f_i(x) + \|A_i x\|_{a(i)} + \sum_{j\in J_0} v_j [G_j(x) + \|C_j x\|_{c(j)}] \Big\}$$
$$-[N_i^\circ(y,\alpha) + \Lambda_0^\circ(y,v,w,\gamma)][g_i(x) - \|B_i x\|_{b(i)}] \Big\}$$
$$\text{(by (5.11) and (5.12))}$$
$$\leqq \sum_{i=1}^{p} u_i \Big\{ D_i^\circ(y,\beta)[f_i(x) + \|A_i x\|_{a(i)}]$$
$$-[N_i^\circ(y,\alpha) + \Lambda_0^\circ(y,v,w,\gamma)][g_i(x) - \|B_i x\|_{b(i)}] \Big\}$$
$$\text{(by the primal feasibility of } x\text{).}$$

Since $u > 0$, the above inequality implies that

$$\Big(D_1^\circ(y,\beta)[f_1(x) + \|A_1 x\|_{a(1)}] - [N_1^\circ(y,\alpha) + \Lambda_0^\circ(y,v,w,\gamma)]$$
$$\times [g_1(x) - \|B_1 x\|_{b(1)}], \cdots, D_p^\circ(y,\beta)[f_p(x) + \|A_p x\|_{a(p)}]$$
$$-[N_p^\circ(y,\alpha) + \Lambda_0^\circ(y,v,w,\gamma)][g_p(x) - \|B_p x\|_{b(p)}] \Big) \nleq (0,\ldots,0),$$

which in turn implies that

$$\varphi(x) = \left(\frac{f_1(x) + \|A_1 x\|_{a(1)}}{g_1(x) - \|B_1 x\|_{b(1)}}, \ldots, \frac{f_p(x) + \|A_p x\|_{a(p)}}{g_p(x) - \|B_p x\|_{b(p)}} \right)$$
$$\nleq \left(\frac{N_1^\circ(y,\alpha) + \Lambda_0^\circ(y,v,w,\gamma)}{D_1^\circ(y,\beta)}, \ldots, \frac{N_p^\circ(y,\alpha) + \Lambda_0^\circ(y,v,w,\gamma)}{D_p^\circ(y,\beta)} \right) = \xi(z).$$

(b) Proceeding as in the proof of part (a), we see that (ii) leads to the following inequality:

$$\Big\langle \sum_{t=1}^{m} \Big\{ \sum_{j\in J_t} v_j [\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k\in K_t} w_k \nabla H_k(y) \Big\}, \eta(x,y) \Big\rangle$$
$$\leqq -\sum_{t=1}^{m} \tilde{\rho}_t \|x-y\|^2.$$

Combining this inequality with (5.13) and using (iii), we obtain

$$\Big\langle \sum_{i=1}^{p} u_i \Big\{ D_i^\circ(y,\beta) \Big\{ \nabla f_i(y) + A_i^T \alpha^i] + \sum_{j\in J_0} v_j [\nabla G_j(y) + C_j^T \gamma^j]$$
$$+ \sum_{k\in K_0} w_k \nabla H_k(y) \Big\}$$
$$-[N_i^\circ(y,\alpha) + \Lambda_0^\circ(y,v,w,\gamma)][\nabla g_i(y) - B_i^T \beta^i] \Big\}, \eta(x,y) \Big\rangle \geqq -\bar{\rho}\|x-y\|^2,$$

which by virtue of (i) implies that $\Phi(x, y, u, u, v, w, \alpha, \beta, \gamma) \geqq \Phi(y, y, u, u, v, w, \alpha, \beta, \gamma) = 0$, where the equality follows from the definitions of $D_i^\circ(y, \beta)$, $N_i^\circ(y, \alpha)$, $i \in \underline{p}$, and $\Lambda_0^\circ(y, v, w, \gamma)$. The rest of the proof is identical to that of part (a).

(c)–(d) The proofs are similar to those of parts (a) and (b). □

THEOREM 5.2 (Strong Duality). *Let x^* be a normal efficient solution of (P) and assume that any one of the four sets of conditions set forth in Theorem 5.1 is satisfied for all feasible solutions of (DIII). Then there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$,* and *$\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that $z^* \equiv (x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is an optimal solution of (DIII) and $\varphi(x^*) = \xi(z^*)$.*

Proof. Since x^* is a normal efficient solution of (P), by Theorem 2.1, there exist $u^*, \alpha^{*i}, \beta^{*i}$, and γ^{*j}, as specified above, and $\bar{v} \in \mathbb{R}^q_+$ and $\bar{w} \in \mathbb{R}^r$ such that (5.11), (5.12), and the following equations hold:

$$\sum_{i=1}^{p} u_i^* \{D_i^\circ(y, \beta)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i^\circ(y, \alpha)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\}$$
$$+ \sum_{j=1}^{q} \bar{v}_j [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^{r} \bar{w}_k \nabla H_k(x^*) = 0, \tag{5.16}$$

$$\bar{v}_j [G_j(x^*) + \|C_j x^*\|_{c(j)}] = 0, \quad j \in \underline{q}, \tag{5.17}$$

$$\langle \alpha^{*i}, A_i x^* \rangle = \|A_i x^*\|_{a(i)}, \quad \langle \beta^{*i}, B_i x^* \rangle = \|B_i x^*\|_{b(i)}, \quad i \in \underline{p}, \tag{5.18}$$

$$\langle \gamma^{*j}, C_j x^* \rangle = \|C_j x^*\|_{c(j)}, \quad j \in \underline{q}. \tag{5.19}$$

Since $c \equiv \sum_{i=1}^{p} u_i^* D_i^\circ(y, \beta) > 0$, $\Lambda_0^\circ(x^*, \bar{v}/c, \bar{w}/c, \gamma^*) = 0$, and (5.19) holds, (5.16) and (5.17) can be expressed as follows:

$$\sum_{i=1}^{p} u_i^* \Big\{ D_i^\circ(y, \beta) \Big\{ \nabla f_i(x^*) + A_i^T \alpha^{*i} + \sum_{j \in J_0} (\bar{v}_j / c)[\nabla G_j(x^*) + C_j^T \gamma^{*j}]$$
$$+ \sum_{k \in K_0} (\bar{w}_k / c) \nabla H_k(x^*)] \Big\} - [N_i^\circ(y, \alpha) + \Lambda_0^\circ(x^*, \bar{v}/c, \bar{w}/c, \gamma^*)]$$
$$\times [\nabla g_i(x^*) - B_i^T \beta^{*i}] \Big\} + \sum_{j \in \underline{q} \setminus J_0} \bar{v}_j [\nabla G_j(x^*) + C_j^T \gamma^{*j}]$$
$$+ \sum_{k \in \underline{r} \setminus K_0} \bar{w}_k \nabla H_k(x^*) = 0, \tag{5.20}$$

$$(\bar{v}_j / c)[G_j(x^*) + \langle \gamma^{*j}, C_j x^* \rangle] = 0, \quad j \in \underline{q}. \tag{5.21}$$

Now, if we let $v_j^* = \bar{v}_j/c$ for each $j \in J_0$, $v_j^* = \bar{v}_j$ for each $j \in \underline{q} \backslash J_0$, $w_k^* = \bar{w}_k/c$ for each $k \in K_0$, $w_k^* = \bar{w}_k$ for each $k \in \underline{r} \backslash K_0$, then from (5.11), (5.12), (5.20), and (5.21) we see that z^* is a feasible solution of (*DIII*), and from (5.18) it follows that $\varphi(x^*) = \xi(z^*)$. That this solution is efficient for (*DIII*) can be verified as in the proof of Theorem 3.2. □

THEOREM 5.3 (Strict Converse Duality). *Let x^* be an efficient solution of* (*P*) *and let $\tilde{z} \equiv (\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ be a feasible solution of* (*DIII*) *such that*

$$\sum_{i=1}^{p} \tilde{u}_i \{ D_i^{\circ}(\tilde{x}, \tilde{\beta})[f_i(x^*) + \|A_i x^*\|_{a(i)}] - [N_i^{\circ}(\tilde{x}, \tilde{\alpha}) + \Lambda_0^{\circ}(\tilde{x}, \tilde{v}, \tilde{w}, \tilde{\gamma})] \times [g_i(x^*) - \|B_i x^*\|_{b(i)}] \} \leqq 0. \tag{5.22}$$

Furthermore, assume that any one of the following four sets of conditions holds:

(a) *The assumptions specified in part* (a) *of Theorem 5.1 are satisfied for the feasible solution $\tilde{z}$ of* (*DIII*), *and the function $\Phi(\cdot, \tilde{x}, \tilde{u}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is $(\eta, \bar{\rho})$-quasiinvex at $\tilde{x}$.*
(b) *The assumptions specified in part* (b) *of Theorem 5.1 are satisfied for the feasible solution $\tilde{z}$ of* (*DIII*), *and the function $\Phi(\cdot, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is strictly $(\eta, \bar{\rho})$-pseudoinvex at $\tilde{x}$.*
(c) *The assumptions specified in part* (c) *of Theorem 5.1 are satisfied for the feasible solution $\tilde{z}$ of* (*DIII*), *and the function $\Phi(\cdot, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is $(\eta, \bar{\rho})$-quasiinvex at $\tilde{x}$.*
(d) *The assumptions specified in part* (d) *of Theorem 5.1 are satisfied for the feasible solution $\tilde{z}$ of* (*DIII*), *and the function $\Phi(\cdot, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ is $(\eta, \bar{\rho})$-quasiinvex at $\tilde{x}$.*

Then $\tilde{x} = x^*$ and $\varphi(x^*) = \xi(\tilde{z})$.

Proof. Suppose to the contrary that $\tilde{x} \neq x^*$. Proceeding as in the proof of part (a) of Theorem 5.1 (with x replaced by x^* and z by $\tilde{z}$), we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i \Big\{ D_i^{\circ}(\tilde{x}, \tilde{\beta})[f_i(x^*) + \|A_i x^*\|_{a(i)}] - [N_i^{\circ}(\tilde{x}, \tilde{\alpha}) + \Lambda_0^{\circ}(\tilde{x}, \tilde{v}, \tilde{w}, \tilde{\gamma})] \times [g_i(x^*) - \|B_i x^*\|_{b(i)}] \Big\} > 0,$$

which contradicts (5.22). Hence, $\tilde{x} = x^*$.

(b)–(d) The proofs are similar to that of part (a). □

THEOREM 5.4 (Weak Duality). *Let x and $z \equiv (y, v, w, \alpha, \beta, \gamma)$ be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) *for each $i \in \underline{p}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at y;*
(ii) *for each $t \in \underline{m}$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is $(\eta, \tilde{\rho}_t)$-quasiinvex at y;*
(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;

(b) (i) *for each $i \in \underline{p}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at y;*
(ii) *for each $t \in \underline{m}$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is strictly $(\eta, \tilde{\rho}_t)$-pseudoinvex at y;*
(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;

(c) (i) *for each $i \in \underline{p}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at y;*
(ii) *for each $t \in \underline{m}$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is $(\eta, \tilde{\rho}_t)$-quasiinvex at y;*
(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t > 0$;

(d) (i) *for each $i \in \underline{p_1}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at y, and for each $i \in \underline{p_2}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at y, where $\{\underline{p_1}, \underline{p_2}\}$ is a partition of $\underline{p}$;*
(ii) *for each $t \in \underline{m}$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is strictly $(\eta, \tilde{\rho}_t)$-pseudoinvex at y;*
(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;

(e) (i) *for each $i \in \underline{p_1} \neq \emptyset$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at y, and for each $i \in \underline{p_2}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at y, where $\{\underline{p_1}, \underline{p_2}\}$ is a partition of $\underline{p}$;*
(ii) *for each $t \in \underline{m}$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is $(\eta, \tilde{\rho}_t)$-quasiinvex at y;*
(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;

(f) (i) *for each $i \in \underline{p}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at y;*
(ii) *for each $t \in \underline{m}_1 \neq \emptyset$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is strictly $(\eta, \tilde{\rho}_t)$-pseudoinvex at y, and for each $t \in \underline{m}_2$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is $(\eta, \tilde{\rho}_t)$-quasiinvex at y, where $\{\underline{m}_1, \underline{m}_2\}$ is a partition of $\underline{m}$;*
(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;

(g) (i) *for each $i \in \underline{p_1}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at y, and for each $i \in \underline{p_2}$, $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at y, where $\{\underline{p_1}, \underline{p_2}\}$ is a partition of $\underline{p}$;*
(ii) *for each $t \in \underline{m}_1$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is strictly $(\eta, \tilde{\rho}_t)$-pseudoinvex at y, and for each $t \in \underline{m}_2$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is $(\eta, \tilde{\rho}_t)$-quasiinvex at y, where $\{\underline{m}_1, \underline{m}_2\}$ is a partition of $\underline{m}$;*
(iii) $\sum_{i=1}^{p} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;
(iv) $\underline{p_1} \neq \emptyset$, $\underline{m_1} \neq \emptyset$, *or* $\sum_{i=1}^{p} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t > 0$.

Then $\varphi(x) \nleq \xi(z)$.

Proof. (a) Suppose to the contrary that $\varphi(x) \leqslant \xi(z)$. This implies that

$$\frac{f_i(x) + \|A_i x\|_{a(i)}}{g_i(x) - \|B_i x\|_{b(i)}} \leqq \frac{f_i(y) + \langle \alpha^i, A_i y\rangle + \Lambda_0^\circ(y, v, w, \gamma)}{g_i(y) - \langle \beta^i, B_i y\rangle} \text{ for each } i \in \underline{p},$$

and

$$\frac{f_m(x) + \|A_m x\|_{a(m)}}{g_m(x) - \|B_m x\|_{b(m)}} < \frac{f_m(y) + \langle \alpha^m, A_m y\rangle + \Lambda_0^\circ(y, v, w, \gamma)}{g_m(y) - \langle \beta^m, B_m y\rangle} \text{ for some } m \in \underline{p}.$$

These inequalities imply that

$$\begin{aligned} &D_i^\circ(y, \beta)[f_i(x) + \|A_i x\|_{a(i)}] - [N_i^\circ(y, \alpha) + \Lambda_0^\circ(y, v, w, \gamma)] \\ &\quad \times [g_i(x) - \|B_i x\|_{b(i)}] \leqq 0 \quad \text{for each } i \in \underline{p}, \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} &D_m^\circ(y, \beta)[f_m(x) + \|A_m x\|_{a(m)}] - [N_m^\circ(y, \alpha) + \Lambda_0^\circ(y, v, w, \gamma)] \\ &\quad \times [g_m(x) - \|B_m x\|_{b(m)}] < 0 \text{ for some } m \in \underline{p}. \end{aligned} \tag{5.24}$$

Keeping in mind that $D_i^\circ(y, \beta) > 0$, $N_i^\circ(y, \alpha) \geqq 0$, $i \in \underline{p}$, and $v \geqq 0$, we see that

$$\begin{aligned} &\Phi_i(x, y, v, w, \alpha, \beta, \gamma) \\ &\quad = D_i^\circ(y, \beta)\Big\{ f_i(x) + \langle \alpha^i, A_i x\rangle + \sum_{j \in J_0} v_j [G_j(x) + \langle \gamma^j, C_j x\rangle] \\ &\qquad + \sum_{k \in K_0} w_k H_k(x)\Big\} - [N_i^\circ(y, \alpha) + \Lambda_0^\circ(y, v, w, \gamma)][g_i(x) - \langle \beta^i, B_i x\rangle] \\ &\quad \leqq D_i^\circ(y, \beta)\Big\{ f_i(x) + \|\alpha^i\|_{a(i)}^* \|A_i x\|_{a(i)} \\ &\qquad + \sum_{j \in J_0} v_j [G_j(x) + \|\gamma^j\|_{c(j)}^* \|C_j x\|_{c(j)}]\Big\} \\ &\qquad - [N_i^\circ(y, \alpha) + \Lambda_0^\circ(y, v, w, \gamma)][g_i(x) - \|\beta^i\|_{b(i)}^* \|B_i x\|_{b(i)}] \\ &\qquad \text{(by Lemma 3.1, definition of } \Lambda_0^\circ(y, v, w, \gamma), \\ &\qquad \text{(5.10), and primal feasibility of } x) \\ &\quad \leqq D_i^\circ(y, \beta)\Big\{ f_i(x) + \|A_i x\|_{a(i)}] + \sum_{j \in J_0} v_j [G_j(x) + \|C_j x\|_{c(j)}]\Big\} \\ &\qquad - [N_i^\circ(y, \alpha) + \Lambda_0^\circ(y, v, w, \gamma)][g_i(x) \\ &\qquad - \|B_i x\|_{b(i)}] \quad \text{(by (5.11) and (5.12))} \\ &\quad \leqq D_i^\circ(y, \beta)[f_i(x) + \|A_i x\|_{a(i)}] - [N_i^\circ(y, \alpha) + \Lambda_0^\circ(y, v, w, \gamma)] \\ &\qquad \times [g_i(x) - \|B_i x\|_{b(i)}] \text{ (by the primal feasibility of } x) \end{aligned}$$

$$\leqq 0 \quad \text{(by (5.23) and (5.24))}$$
$$= D_i^{\circ}(y,\beta)\Big\{f_i(y)+\langle\alpha^i, A_i y\rangle] + \sum_{j\in J_0} v_j[G_j(y)+\langle\gamma^j, C_j y\rangle]$$
$$+\sum_{k\in K_0} w_k H_k(y)\Big\} - [N_i^{\circ}(y,\alpha)+\Lambda_0^{\circ}(y,v,w,\gamma)][g_i(y)-\langle\beta^i, B_i y\rangle]$$
(by the definitions of $D_i^{\circ}(y,\beta)$, $N_i^{\circ}(y,\alpha)$, $i\in \underline{p}$, and $\Lambda_0^{\circ}(y,v,w,\gamma)$)
$$=\Phi_i(y,y,v,w,\alpha,\beta,\gamma),$$

which in view of (i) implies that for each $i\in\underline{p}$,

$$\Big\langle D_i^{\circ}(y,\beta)[\nabla f_i(y)+A_i^T\alpha^i] + \sum_{j\in J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j] + \sum_{k\in K_0} w_k \nabla H_k(y)$$
$$-[N_i^{\circ}(y,\alpha)+\Lambda_0^{\circ}(y,v,w,\gamma)][\nabla g_i(y)-B_i^T\beta^i], \eta(x,y)\Big\rangle < -\bar{\rho}_i\|x-y\|^2.$$

Since $u>0$, the above inequalities yield

$$\Big\langle \sum_{i=1}^p u_i\Big\{D_i^{\circ}(y,\beta)\Big\{\nabla f_i(y)+A_i^T\alpha^i + \sum_{j\in J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j]$$
$$+\sum_{k\in K_0} w_k\nabla H_k(y)\Big\} - [N_i^{\circ}(y,\alpha)+\Lambda_0^{\circ}(y,v,w,\gamma)]$$
$$\times[\nabla g_i(y)-B_i^T\beta^i]\Big\}, \eta(x,y)\Big\rangle < -\sum_{i=1}^p u_i\bar{\rho}_i\|x-y\|^2. \tag{5.25}$$

As seen in the proof of Theorem 5.1, our assumptions in (ii) lead to

$$\Big\langle \sum_{t=1}^m \Big\{\sum_{j\in J_t} v_j[\nabla G_j(y)+C_j^T\gamma^j] + \sum_{k\in K_t} w_k\nabla H_k(y)\Big\}, \eta(x,y)\Big\rangle \leqq$$
$$-\sum_{t=1}^m \tilde{\rho}_t\|x-y\|^2,$$

which when combined with (5.13), results into

$$\Big\langle \sum_{i=1}^p u_i\Big\{D_i^{\circ}(y,\beta)\Big\{\nabla f_i(y)+A_i^T\alpha^i + \sum_{j\in J_0} v_j[\nabla G_j(y)+C_j^T\gamma^j]$$
$$+\sum_{k\in K_0} w_k\nabla H_k(y)\Big\} - [N_i^{\circ}(y,\alpha)+\Lambda_0^{\circ}(y,v,w,\gamma)][\nabla g_i(y)-B_i^T\beta^i]\Big\},$$
$$\times\eta(x,y)\Big\rangle \geqq \sum_{t=1}^m \tilde{\rho}_t\|x-y\|^2.$$

In view of (iii), this inequality contradicts (5.25). Hence, $\varphi(x) \nleq \xi(z)$.

(b)–(g) The proofs are similar to that of part (a). □

THEOREM 5.5 (Strong Duality). *Let x^* be a normal efficient solution of (P) and assume that any one of the seven sets of conditions set forth in Theorem 5.4 is satisfied for all feasible solutions of (DIII). Then there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that $z^* \equiv (x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is an efficient solution of (DIII) and $\varphi(x^*) = \xi(z^*)$.*

Proof. The proof is similar to that of Theorem 5.2. □

As pointed out earlier, the generalized duality models discussed in this section contain numerous interesting and important special cases which can readily be identified by appropriate choices of the partitioning sets $J_0, J_1, \ldots, J_m, K_0, K_1, \ldots,$ and K_m, and the arbitrary norms $\|\cdot\|_{a(i)}$, $\|\cdot\|_{b(i)}$, $i \in \underline{p}$, and $\|\cdot\|_{c(j)}$, $j \in \underline{q}$.

6. Duality Model IV

In this section we discuss four additional duality models for (P) which are different from those presented in the preceding sections. In these duality formulations we utilize a partition of $\underline{p}$ in addition to those of $\underline{q}$ and $\underline{r}$. This particular partitioning method was used previously for a special case of (P) in [19]. In our duality theorems, we impose appropriate generalized (η, ρ)-invexity requirements on certain combinations of the functions $\mathcal{E}_i(\cdot, y, \alpha, \beta)$, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$.

Let $\{I_0, I_1, \ldots, I_\ell\}$ be a partition of $\underline{p}$ such that $L = \{0, 1, 2, \ldots, \ell\} \subset M = \{0, 1, \ldots, m\}$, and let the function $\Pi_t(\cdot, \bar{x}, u, v, w, \alpha, \beta, \gamma) : X \to \mathbb{R}$ be defined, for fixed $\bar{x}, u, v, w, \alpha, \beta$, and γ, by

$$\begin{aligned}\Pi_t(x, \bar{x}, u, v, w, \alpha, \beta, \gamma) = &\sum_{i \in I_t} u_i\{D_i^\circ(\bar{x}, \beta)[f_i(x) + \langle \alpha^i, A_i x\rangle]\\ &- N_i^\circ(\bar{x}, \alpha)[g_i(x) - \langle \beta^i, B_i x\rangle]\}\\ &+ \sum_{j \in J_t} v_j[G_j(x) + \langle \gamma^j, C_j x\rangle]\\ &+ \sum_{k \in K_t} w_k H_k(x), \quad t \in \underline{m}.\end{aligned}$$

Consider the following dual problems:

$$\text{(CIV)} \quad \text{Maximize} \left(\frac{f_1(y) + \|A_1 y\|_{a(1)}}{g_1(y) - \|B_1 y\|_{b(1)}}, \ldots, \frac{f_p(y) + \|A_p y\|_{a(p)}}{g_p(y) - \|B_p y\|_{b(p)}}\right)$$

subject to (3.1), (3.3)–(3.7), and

$$\sum_{j \in J_t} v_j[G_j(y) + \|C_j y\|_{c(j)}] + \sum_{k \in K_t} w_k H_k(y) \geqq 0, \quad t \in M; \tag{6.1}$$

$$(\tilde{\mathrm{C}}\mathrm{IV}) \quad \text{Maximize}\left(\frac{f_1(y) + \|A_1 y\|_{a(1)}}{g_1(y) - \|B_1 y\|_{b(1)}}, \dots, \frac{f_p(y) + \|A_p y\|_{a(p)}}{g_p(y) - \|B_p y\|_{b(p)}}\right)$$

subject to (3.3)–(3.8) and (6.1);

$$(\mathrm{DIV}) \quad \text{Maximize}\left(\frac{f_1(y) + \langle \alpha^1, A_1 y\rangle}{g_1(y) - \langle \beta^1, B_1 y\rangle}, \dots, \frac{f_p(y) + \langle \alpha^p, A_p y\rangle}{g_p(y) - \langle \beta^p, B_p y\rangle}\right)$$

subject to

$$\sum_{i=1}^{p} u_i\{D_i^\circ(y,\beta)[\nabla f_i(y) + A_i^T \alpha^i] - N_i^\circ(y,\alpha)[\nabla g_i(y) - B_i^T \beta^i]\}$$
$$+ \sum_{j=1}^{q} v_j[\nabla G_j(y) + C_j^T \gamma^j] + \sum_{k=1}^{r} w_k \nabla H_k(y) = 0, \tag{6.2}$$

$$\sum_{j \in J_t} v_j[G_j(y) + \langle \gamma^j, C_j y\rangle] + \sum_{k \in K_t} w_k H_k(y) \geqq 0, \quad t \in M, \tag{6.3}$$

$$\|\alpha^i\|_{a(i)}^* \leqq 1, \qquad \|\beta^i\|_{b(i)}^* \leqq 1, \quad i \in \underline{p}, \tag{6.4}$$

$$\|\gamma^j\|_{c(j)}^* \leqq 1, \quad j \in \underline{q}, \tag{6.5}$$

$$y \in X,\ u \in U,\ v \in \mathbb{R}_+^q,\ w \in \mathbb{R}^r,\ \alpha^i \in \mathbb{R}^{\ell_i},\ \beta^i \in \mathbb{R}^{m_i},\ i \in \underline{p},\ \gamma^j \in \mathbb{R}^{n_j},\ j \in \underline{q}; \tag{6.6}$$

$$(\tilde{\mathrm{D}}\mathrm{IV}) \quad \text{Maximize}\left(\frac{f_1(y) + \langle \alpha^1, A_1 y\rangle}{g_1(y) - \langle \beta^1, B_1 y\rangle}, \dots, \frac{f_p(y) + \langle \alpha^p, A_p y\rangle}{g_p(y) - \langle \beta^p, B_p y\rangle}\right)$$

subject to (3.14) and (6.3)–(6.6).

The remarks and observations made earlier about the relationships among (CI), ($\tilde{\mathrm{C}}$I), (*DI*), and ($\tilde{\mathrm{D}}$I) are, of course, also valid for (CIV), ($\tilde{\mathrm{C}}$IV), (DIV), and ($\tilde{\mathrm{D}}$IV). As in the preceding sections, we shall work with the reduced versions (DIV) and ($\tilde{D}$IV), and, in particular, consider the pair (P)–(DIV).

The next two theorems show that (DIV) is a dual problem for (P).

THEOREM 6.1 (Weak Duality). *Let x and $z \equiv (y, u, v, w, \alpha, \beta, \gamma)$ be arbitrary feasible solutions of (P) and (DIV), respectively, and assume that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) *for each $t \in L$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is strictly (η, ρ_t)-pseudoinvex at y;*
(ii) *for each $t \in M \setminus L$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is (η, ρ_t)-quasiinvex at y;*
(iii) $\sum_{t \in M} \rho_t \geqq 0$;

(b) (i) *for each $t \in L$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is prestrictly (η, ρ_t)-quasiinvex at y;*
(ii) *for each $t \in M \setminus L$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is strictly (η, ρ_t)-pseudoinvex at y;*
(iii) $\sum_{t \in M} \rho_t \geqq 0$;

(c) (i) *for each $t \in L$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is prestrictly (η, ρ_t)-quasiinvex at y;*
(ii) *for each $t \in M \setminus L$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is (η, ρ_t)-quasiinvex at y;*
(iii) $\sum_{t \in M} \rho_t > 0$;

(d) (i) *for each $t \in L_1$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is strictly (η, ρ_t)-pseudoinvex at y, and for each $t \in L_2$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is prestrictly (η, ρ_t)-quasiinvex at y, where $\{L_1, L_2\}$ is a partition of L;*
(ii) *for each $t \in M \setminus L$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is strictly (η, ρ_t)-pseudoinvex at y;*
(iii) $\sum_{t \in M} \rho_t \geqq 0$;

(e) (i) *for each $t \in L_1 \neq \emptyset$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is strictly (η, ρ_t)-pseudoinvex at y, and for each $t \in L_2$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is prestrictly (η, ρ_t)-quasiinvex at y, where $\{L_1, L_2\}$ is a partition of L;*
(ii) *for each $t \in M \setminus L$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is (η, ρ_t)-quasiinvex at y;*
(iii) $\sum_{t \in M} \rho_t \geqq 0$;

(f) (i) *for each $t \in L$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is prestrictly (η, ρ_t)-quasiinvex at y;*
(ii) *for each $t \in (M \setminus L)_1 \neq \emptyset$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is strictly (η, ρ_t)-pseudoinvex at y, and for each $t \in (M \setminus L)_2$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is (η, ρ_t)-quasiinvex at y, where $\{(M \setminus L)_1, (M \setminus L)_2\}$ is a partition of $M \setminus L$;*
(iii) $\sum_{t \in M} \rho_t \geqq 0$;

(g) (i) *for each $t \in L_1$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is strictly (η, ρ_t)-pseudoinvex at y, and for each $t \in L_2$, $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma)$ is prestrictly (η, ρ_t)-quasiinvex at y, where $\{L_1, L_2\}$ is a partition of L;*
(ii) *for each $t \in (M \setminus L)_1$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is strictly (η, ρ_t)-pseudoinvex at y, and for each $t \in (M \setminus L)_2$, $\Lambda_t^{\circ}(\cdot, v, w, \gamma)$ is (η, ρ_t)-quasiinvex at y, where $\{(M \setminus L)_1, (M \setminus L)_2\}$ is a partition of $M \setminus L$;*
(iii) $\sum_{t \in M} \rho_t \geqq 0$;
(iv) $L_1 \neq \emptyset$, $(M \setminus L)_1 \neq \emptyset$, *or* $\sum_{t \in M} \rho_t > 0$.

Then $\varphi(x) \nleq \omega(z)$, where $\omega = (\omega_1, \ldots, \omega_p)$ is the objective function of (DIV).

Proof. (a) Suppose to the contrary that $\varphi(x) \leqslant \omega(z)$. As shown in the proof of Theorem 4.6, this supposition leads to the following inequalities:

$$D_i^\circ(y,\beta)[f_i(x)+\|A_ix\|_{a(i)}]-N_i^\circ(y,\alpha)[g_i(x)-\|B_ix\|_{b(i)}]\leqq 0, \quad i\in \underline{p},$$

with strict inequality holding for at least one index $m \in \underline{p}$. Therefore, for each $t \in L$, we have

$$\sum_{i\in I_t} u_i\{D_i^\circ(y,\beta)[f_i(x)+\|A_ix\|_{a(i)}]-N_i^\circ(y,\alpha)[g_i(x)-\|B_ix\|_{b(i)}]\}\leqq 0. \tag{6.7}$$

Since

$$\begin{aligned}
&\Pi_t(x,y,u,v,w,\alpha,\beta,\gamma)\\
&\quad=\sum_{i\in I_t} u_i\{D_i^\circ(y,\beta)[f_i(x)+\langle\alpha^i,A_ix\rangle]-N_i^\circ(y,\alpha)[g_i(x)-\langle\beta^i,B_ix\rangle]\}\\
&\qquad+\sum_{j\in J_t} v_j[G_j(x)+\langle\gamma^j,C_jx\rangle]+\sum_{k\in K_t} w_kH_k(x)\\
&\quad\leqq\sum_{i\in I_t} u_i\{D_i^\circ(y,\beta)[f_i(x)+\|\alpha^i\|^*_{a(i)}\|A_ix\|_{a(i)}]\\
&\qquad-N_i^\circ(y,\alpha)[g_i(x)-\|\beta^i\|^*_{b(i)}\|B_ix\|_{b(i)}]\}\\
&\qquad+\sum_{j\in J_t} v_j[G_j(x)+\|\gamma^j\|^*_{c(j)}\|C_jx\|_{c(j)}]\\
&\qquad\text{(by Lemma 3.1 and primal feasibility of } x)\\
&\quad\leqq\sum_{i\in I_t} u_i\{D_i^\circ(y,\beta)[f_i(x)+\|A_ix\|_{a(i)}]-N_i^\circ(y,\alpha)[g_i(x)-\|B_ix\|_{b(i)}]\}\\
&\qquad+\sum_{j\in J_t} v_j[G_j(x)+\|C_jx\|_{c(j)}]\ \text{(by (6.4) and (6.5))}\\
&\quad\leqq\sum_{i\in I_t} u_i\{D_i^\circ(y,\beta)[f_i(x)+\|A_ix\|_{a(i)}]-N_i^\circ(y,\alpha)[g_i(x)-\|B_ix\|_{b(i)}]\}\\
&\qquad\text{(by the primal feasibility of } x)\\
&\quad\leqq 0 \quad \text{(by (6.7))}\\
&\quad\leqq\sum_{i\in I_t} u_i\{D_i^\circ(y,\beta)[f_i(y)+\langle\alpha^i,A_iy\rangle]-N_i^\circ(y,\alpha)[g_i(y)-\langle\beta^iB_iy\rangle]\}\\
&\qquad+\sum_{j\in J_t} v_j[G_j(y)+\langle\gamma^j,C_jy\rangle]+\sum_{k\in K_t} w_kH_k(y)\\
&\qquad\text{(by the definitions of } D_i^\circ(y,\beta) \text{ and } N_i^\circ(y,\alpha),\ i\in\underline{p}, \text{ and (6.3))}\\
&\quad=\Pi_t(y,y,u,v,w,\alpha,\beta,\gamma),
\end{aligned}$$

it follows from (i) that

$$\Big\langle \sum_{i\in I_t} u_i\{D_i^\circ(y,\beta)[\nabla f_i(y)+A_i^T\alpha^i]-N_i^\circ(y,\alpha)[\nabla g_i(y)-B_i^T\beta^i]\} + \sum_{j\in J_t} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_t} w_k\nabla H_k(y),\eta(x,y)\Big\rangle < -\rho_t\|x-y\|^2.$$

Adding the above inequalities, we get

$$\Big\langle \sum_{i=1}^{p} u_i\{D_i^\circ(y,\beta)[\nabla f_i(y)+A_i^T\alpha^i]-N_i^\circ(y,\alpha)[\nabla g_i(y)-B_i^T\beta^i]\} + \sum_{t\in L}\Big\{\sum_{j\in J_t} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_t} w_k\nabla H_k(y)\Big\},\eta(x,y)\Big\rangle < -\sum_{t\in L}\rho_t\|x-y\|^2. \tag{6.8}$$

As shown in the proof of Theorem 5.1, for each $t\in M\backslash L$, $\Lambda_t^\circ(x,v,w,\gamma)\leqq \Lambda_t^\circ(y,v,w,\gamma)$, which in view of (ii) implies that

$$\Big\langle \sum_{j\in J_t} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_t} w_k\nabla H_k(y),\eta(x,y)\Big\rangle \leqq -\rho_t\|x-y\|^2.$$

Summing over t, we obtain

$$\Big\langle \sum_{t\in M\backslash L}\Big\{\sum_{j\in J_t} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k\in K_t} w_k\nabla H_k(y)\Big\},\eta(x,y)\Big\rangle \leqq -\sum_{t\in M\backslash L}\rho_t\|x-y\|^2. \tag{6.9}$$

Now combining (6.8) and (6.9) and using (iii), we obtain

$$\Big\langle \sum_{i=1}^{p} u_i\{D_i^\circ(y,\beta)[\nabla f_i(y)+A_i^T\alpha^i]-N_i^\circ(y,\alpha)[\nabla g_i(y)-B_i^T\beta^i]\} + \sum_{j=1}^{q} v_j[\nabla G_j(y)+C_j^T\gamma^j]+\sum_{k=1}^{r} w_k\nabla H_k(y),\eta(x,y)\Big\rangle < -\sum_{t=1}^{m}\rho_t\|x-y\|^2\leqq 0,$$

which contradicts (6.2). Therefore, $\varphi(x)\nleq\omega(z)$.

(b)–(g) The proofs are similar to that of part (a). □

THEOREM 6.2 (Strong Duality). *Let x^* be a normal efficient solution of (P) and assume that any one of the seven sets of conditions set forth in Theorem* 6.1 *is satisfied for all feasible solutions of (DIV). Then there exist* $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, *and* $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, *such that* $z^* \equiv (x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is an efficient solution of (DIV) and* $\varphi(x^*) = \omega(z^*)$.

Proof. The proof is similar to that of Theorem 3.2. □

Evidently, the four duality models presented in this section contain a multitude of important special cases which can easily be generated by appropriate choices of the partitioning sets. They collectively subsume a variety of existing dual problems and include a number of new duality formulations for several classes of single- and multiple-objective nonlinear programming problems.

7. Concluding Remarks

In this paper, we have established a fairly large number of semiparametric duality results under a variety of generalized (η, ρ)-invexity assumptions for a multiobjective fractional programming problem containing arbitrary norms (and square roots of positive semidefinite quadratic forms). Each one of these duality results can easily be modified and restated for each one of the ten special cases of the prototype problem (P) designated as (P1) – (P10) in Section 1, and hence they collectively subsume a truly vast number of duality results previously established by different methods for various classes of nonlinear programming problems with multiple, fractional, and conventional objective functions. Furthermore, the style and techniques employed in this paper can be utilized for developing similar results for some other classes of optimization problems involving more general types of convex functions. These include discrete and continuous minmax programming problems, various classes of semiinfinite programming problems, and certain types of continuous-time programming problems.

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